

# COSET SPACE DIMENSIONAL REDUCTION

by

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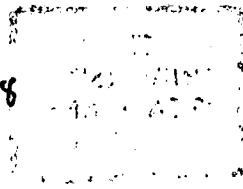
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### DECLARATION

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(J.A. Henderson)

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TO MY FAMILY



# ABSTRACT

In this thesis we investigate two areas of application of the coset space dimensional reduction (CSDR) scheme.

(i) In its earliest applications CSDR was used to obtain Yang-Mills-Higgs theories from pure Yang-Mills theories in higher dimensions. In certain models relationships between the parameters of the four dimensional theory were obtained. We consider the effect of one loop corrections to these models and find that the relationships do not survive beyond the tree level.

(ii) More recently coset space dimensional reduction has found an application in Becchi-Rouet- Stora-Tyutin supersymmetry. An elegant framework for quantisation of gauge fields in which the gauge fixing and compensating ghosts arise automatically is over six-dimensional superspace. Taking the coset space to be  $Sp(2) \wedge T_2 / Sp(2)$  the extended BRST transformations correspond to translations in the extra two coordinates. We apply this to two new cases.

Firstly, we consider rank- $R$  antisymmetric tensor gauge fields. After dimensional reduction we obtain two  $(R-1)$  fermionic ghosts, three  $(R-2)$  bosonic ghosts, ... , down to  $(R+1)$  scalar ghosts. This is the correct ghost spectrum required to formally ensure unitarity of the theory.

Secondly, we covariantly quantise spinor-vector gauge fields in infinite dimensional representations of  $OSp(4/2)$ . After dimensional reduction we find the usual spectrum of Fadeev-Popov and Nielsen-Kallosh ghosts.

Finally, we examine in general the inhomogeneous Grassmann rotation group  $Sp(2) \ltimes T_2$  and its representations which underlie all the above applications. The states can be labelled by pseudomass and pseudospin while the physical state vectors correspond to wave packets over fermionic momentum.

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## 1 INTRODUCTION

The purpose of this introductory chapter is to place the subject matter of this thesis in a historical context. In the first section we discuss progress made towards the goal of unification and in the next section we discuss unitarity. The structure of the thesis is then outlined in Section 1.3.

### 1.1 UNIFICATION

The goal of unification is to find a single theory encompassing all the known forces. Ideally such a theory would reveal some deep connection between the forces while explaining their apparent diversity.

In the 19th century, Maxwell unified the behaviour of electric and magnetic fields in the theory of electromagnetism. It was later realised by Fock [1], London [2] and Weyl [3] that the quantised version of this theory, quantum electrodynamics (Q.E.D.), is invariant under local changes of phase. The associated gauge group is  $U(1)$  and the gauge field is the photon. Einstein's theory of General Relativity is also a local theory. It is invariant under local Poincaré transformations with associated gauge field, the graviton [4]. The success of these two local theories inspired others to search for local theories of the strong and weak interactions.

In 1954, Yang and Mills [5], found that three massless vector gauge fields, (two of which were charged, the other neutral), were needed to ensure invariance under local isotopic transformations. The associated gauge group is  $SU(2)$ . Since isotopic symmetry relates particles with identical isospin, e.g. proton and neutron, they had hoped to obtain a local theory of hadronic interactions. However it was known that these are short range and hence must be mediated by massive gauge fields. The theory was thus an unrealistic one.

In 1961, Glashow [6], proposed a partially symmetric theory unifying the electromagnetic and weak interactions. These electroweak interactions would be mediated by the three Yang-Mills gauge fields which would somehow acquire mass. In addition, he showed that the simplest such theory necessitates the introduction of another neutral vector gauge field. In this scheme the gauge group is  $SU(2) \times U(1)$ , invariant under local isotopic transformations and hypercharge. His proposal did not explain how the vector gauge fields could acquire mass nor whether the theory would be renormalisable. Weinberg [7] and, independently, Salam [8] suggested that the gauge fields could acquire mass by spontaneous symmetry breakdown, which we discuss below, and subsequently, 'tHooft [9] proved the theory was renormalisable.

The idea of spontaneous symmetry breakdown originated in many-body systems such as the ferromagnet and the Bardeen-Cooper-Schrieffer [10] superconductor. The basic idea of SSB is that it is possible for a ground state to be non-invariant under a symmetry possessed by the full Lagrangian. In the case of the ferromagnet the ground state is not invariant under a global rotation of spin. For the superconductor the symmetry 'broken' by the ground state is a local symmetry (see [10] for more details). Goldstone [11] first introduced the idea of a hidden global symmetry to field theories. In [12], Goldstone et al. showed that whenever a global symmetry is hidden massless particles, called Goldstone bosons, occur in the theory. Then Higgs [13], Kibble [14], Brout and Englert [15] showed that if a local symmetry is hidden the Goldstone bosons combine with the zero-helicity states of the gauge fields to give the gauge bosons a mass. This is called the Higgs mechanism. For a review of this topic see Abers and Lee [16].

In the Glashow-Weinberg-Salam, electroweak theory the two charged

vector gauge fields acquire mass and become the  $W^\pm$  particles. The two neutral gauge fields combine with one Higgs boson to become the massive  $Z^0$  and the massless photon. The remaining Higgs should be observable as a massive scalar particle. By measuring the electric charge, the weak angle and the weak coupling constant one can predict the masses of the  $W^\pm$  and  $Z^0$ . However the mass of the Higgs is not determined.

The Standard Model [17] has gauge symmetry  $SU(3) \times SU(2) \times U(1)$ . It incorporates all the results of the electroweak theory and includes  $SU(3)$  the group associated with local colour transformations mediated by the gluons. The Standard Model is not in conflict with any experimental evidence to date however it leaves many problems unsolved. For example (i) inclusion of gravity, (ii) mass of the Higgs, (iii) baryon number asymmetry in the universe, (iv) charge quantisation and (v) parity violation. These and many more problems need to be solved by going beyond the Standard Model while retaining all of its successful features.

In Grand Unified Theories,  $SU(3) \times SU(2) \times U(1)$  is embedded in a larger gauge group. The resultant models give a prediction for the weak angle, incorporate charge quantisation and predict proton decay. These could solve problems (iii) and (iv) above. The simplest G.U.T. has gauge group  $SU(5)$ . This model was proposed by Georgi and Glashow [18]. In its simplest form it has been ruled out on the basis of its prediction for the proton lifetime. For a recent survey of proton lifetime experiments and results see Enqvist and Nanopoulos [19].

Other ideas are compositeness [20] and technicolour [21] in which the quarks are no longer considered as the fundamental building blocks of hadrons. Continuing the study of symmetries, which has



led to the view of all four forces as being described by local theories, we are led to consider supersymmetry [22]. Supersymmetry relates fermions to bosons and vice versa. An interesting observation is that the net result of two supersymmetric transformations is a Poincaré transformation. Since gravity is described by local Poincaré transformations local supersymmetry is called Supergravity and may provide an answer to (i) above. For a review of Supergravity see van Nieuwenhuizen's report [23]. Both Supergravity and Superstring [24] theories involve higher dimensions.

The first successful use of higher dimensions to unify forces was by Kaluza [25] in 1921. By appending a fifth coordinate, which lay on a circle with group  $U(1)$ , to four-dimensional space-time, he was able to unify the theories of gravitation and electromagnetism in 5-dimensional gravity. Under a coordinate transformation in the fifth dimension the  $g_{\mu 5}(X^\mu)$  component of the 5-dimensional gravitational field, underwent a gauge transformation. Einstein's equations in 5 dimensions became Einstein's equations for  $g_{\mu\nu}$ , Maxwell's equations for  $g_{\mu 5}$  and the Klein-Gordon equation for  $g_{55}$ . The size of the circle was related to the gravitational coupling constant and of the order of  $10^{-33}$  cm. Klein's [26] contribution to the Kaluza-Klein theory was to allow the five-dimensional gravitational field to depend also on the fifth coordinate. He then performed a Fourier expansion about this fifth, periodic coordinate. The  $n=0$  modes corresponded to Kaluza's result while the  $n \neq 0$  modes corresponded to massive particles, with masses of the order of the Planck mass, and quantised charge. For an extensive review of Kaluza-Klein type models see Duff et al. [27]. In extended Kaluza-Klein models one takes the extra dimensions to form a compact space with associated group  $G \supset SU(3) \times SU(2) \times U(1)$ . The massless sector

then contains the gravitational field, Yang-Mills gauge fields and a Higgs sector. Apart from the problem of how to compactify the extra dimensions it is found, Witten [28], that Kaluza-Klein models cannot produce parity violation in the four-dimensional model. Thus problem (v) above remains unsolved.

In higher-dimensional theories one can either allow the fields to have a completely arbitrary dependence on the extra coordinates, as above, and perform a harmonic expansion (see Salam and Strathdee [29] for a general discussion of this) or restrict the dependence of the fields in some way. One method is by Legendre transformation [30] in which the hamiltonian in higher dimensions is assumed to be independent of the extra coordinates. Another way is to exploit a symmetry of the action to ensure that the Lagrangian is independent of the extra coordinates. This can be a global symmetry [31] or, in the case of coset space dimensional reduction (CSDR), a local one. In the latter case it has been shown by Palla [32] that these finite modes are not identical to the massless modes of the harmonic expansion.

The CSDR scheme was originated by Forgacs and Manton [33]. Since we give a detailed review of this scheme in Chapter 2 we only briefly state the achievements of the scheme, so far, in this introductory section. For reviews in the literature see [34, 35]. In this scheme a pure Yang-Mills theory in higher dimensions reduces to a Yang-Mills-Higgs theory in four dimensions. The four dimensional theory is completely determined once the gauge group,  $G$ , and coupling constant in higher dimensions have been chosen as well as the coset space  $S/R$ , its size and the embedding of  $R$  in  $G$ . The gauge group in four dimensions is the centraliser of  $R$  in  $G$ . Manton [36] reproduced the bosonic sector of electroweak theory, with predictions for the weak angle and

Higgs mass, from a six-dimensional pure Yang-Mills theory. In reference [37] it was noticed that if  $S$  was also embedded in  $G$  then the group after spontaneous symmetry breaking was determined by group theoretic arguments to be the centraliser of  $S$  in  $G$ . There was no need to explicitly minimise the Higgs potential, a difficult task in practice. Fermions were included in the scheme by Manton [38]. He observed that one could get parity violation in the four dimensional theory if one started with Weyl spinors in a complex representation and  $\text{rank } S = \text{rank } R$ . Another possibility was given in [39]. In a series of papers on model building with fermions Barnes et al. [35, 40, 41] showed that such parity violation did not occur when  $S \subset G$ . It is possible that for stable compactifying solutions of the full Einstein-Yang-Mills theory we must relax the condition  $S \subset G$  [42, 43, 44]. So far most coset spaces studied have been symmetric for convenience of calculation. The authors of [45] have extended the scheme to non-symmetric spaces. Further applications of CSDR have been made to supersymmetry [39, 42], supergravity [46], QCD [47] and superstrings [48].

## 1.2 UNITARITY

In the last section we mentioned that all four forces known at present may be described by local gauge theories. Unfortunately all gauge theories have problems due to the presence of non-physical modes. One cannot construct, in a Lorentz covariant manner, a Lagrangian without unphysical modes such as the non-transverse states of the photon. In order to perform perturbation theory one adds a covariant gauge-fixing term to the Lagrangian to give the physical modes a well-defined propagator. Unfortunately one has then a well-defined propagator for the non-physical modes also. It may then be

feared that these will propagate in virtual, intermediate states and violate unitarity since they can have a negative norm. In order to reconcile Lorentz covariance, non-physical modes and unitarity, it becomes necessary to add ghost terms to the Lagrangian.

Feynman [49] first postulated the existence of extra terms in the Lagrangian whose sole purpose would be to cancel unwanted modes. He constructed modified Feynman rules for one-loop order in non-Abelian theories. These were generalised, with difficulty, to arbitrary loop order by De Witt [50] and Mandelstam [51]. In 1967 Fadeev and Popov [52] gave a compact derivation of the modified rules using a path-integral formalism. In their approach, after adding a covariant gauge-fixing term to the Lagrangian one modifies the functional integral measure by a compensating Jacobian, called the Fadeev-Popov determinant. The ghosts play only a formal role, being introduced as a convenient set of anti-commuting variables which allow one to exponentiate the Fadeev-Popov determinant and obtain a polynomial Lagrangian suitable for perturbation theory. Naive application of this idea led to incorrect ghost terms for antisymmetric tensor fields [53, 54].

In 1974 Becchi, Rouet, Stora [55] and later Tyutin [56] observed that the action including gauge-fixing and ghost terms for QED and Yang-Mills theories is invariant under a transformation which mixes the gauge and ghost fields. These are called the BRST transformations. Kugo and Ojima [57] were then able to construct a canonical, covariant method of proving S-matrix unitarity based on the BRST algebra. This method has been used to prove unitarity for supergravity [58] and higher-rank antisymmetric, tensor field theories [59]. The physical vacuum is defined by  $Q_B|0\rangle = 0$  where  $Q_B$  is the generator of the BRST transformations. Based on the BRST symmetry alternative methods of

introducing the ghosts were proposed. In particular, Ferrara, Piguet and Schweda [60] considered a superfield formalism of QED in which the fields are defined over a five-dimensional, graded space-time. The BRST transformations were then given by translations in the fifth dimension.

Subsequently, the dual BRST, ~~BRST~~, symmetry was discovered [61, 62, 63] in which the roles of the ghosts and antighosts are interchanged. Bonora and Tonin [64] then extended the supersymmetry to include ~~BRST~~ invariance. This was achieved using a six-dimensional space in which the fifth and sixth coordinates are anticommuting. Imposing the condition that the supercurvature should vanish in the Grassmann directions was sufficient for supertranslations in the extra coordinates to correspond to the extended BRST transformations. Delbourgo and Jarvis [65] enlarged the symmetry group to include transformations mixing the ordinary and Grassmann coordinates. The BRST symmetry has been elevated to a general principle for constructing quantised field theories [66]. Research concerned with a satisfactory geometric setting for ghosts has included study of a fibre bundle made with Grassmann variables in the vertical directions [67, 68, 69].

In [70] Delbourgo et al. applied the method of coset space dimensional reduction to the quantisation of vector gauge fields. The four-dimensional action was BRST invariant by construction. Counting rules [71, 72] for  $OSp(n/m)$  tensors suggested that higher-rank, tensor fields may also be quantised in this way although some difficulty is anticipated for spinor-vector fields [72]. In the next section we outline our work in this area.

### 1.3 STRUCTURE OF THE THESIS

In Chapter 2 we give a detailed review of the coset-space, dimensional reduction scheme whose use in model building and BRST quantisation has been mentioned in the previous two sections. We discuss Manton's model [36] as an example of a Yang-Mills-Higgs theory obtained from a pure Yang-Mills theory in higher-dimensions. Then we discuss the model by Delbourgo et al. [70] as an application of CSDR to BRST quantisation.

The appeal of the CSDR scheme to model builders lies in its prediction of relationships between parameters of the four-dimensional theory. In Chapter 3 we investigate whether these relationships, predicted at the classical level, survive one-loop renormalisation corrections. For a wide variety of models we find that the classical predictions are destroyed by quantum effects.

The second application of the CSDR scheme, which has proved successful in the past, is the BRST quantisation of vector gauge fields [70]. In the previous section we mentioned that the quantisation of antisymmetric tensor gauge fields initially caused some difficulty [52, 53]. However counting arguments suggested that the CSDR scheme could be used to successfully quantise these fields [71]. In Chapter 4, therefore, we apply the CSDR method to the BRST quantisation of rank- $R$ , antisymmetric tensor fields and are able to obtain the correct ghost-spectrum and degree-of-freedom count to formally ensure unitarity of the theory.

In Section 1.2 we mentioned that some difficulty was to be expected if the same scheme were applied to spinor-vector fields. If these fields are taken in a finite-dimensional representation of the  $OSp(n/m)$  group, appropriate to BRST quantisation, then we would incorrectly obtain zero degrees of freedom [72]. In Chapter 5 we

apply the CSDR scheme of BRST quantisation to spinor-vectors in an infinite dimensional representation of  $OSp(4/2)$ . After dimensional reduction, for a particular choice of gauge-fixing parameters, we achieve the correct Fadeev-Popov and Nielsen-Kallosh ghost structure for the spinor sector of supergravity.

The Grassmann Euclidean group  $Sp(2) \wedge T_2$  underlies all of our applications of CSDR to BRST quantisation. (Specifically, the unitarity requirement corresponds to focussing on 'physical' representations characterised by vanishing Grassmann momentum). In Chapter 6 we examine this group and its representations in general making contact, where appropriate, with the previous discussion. The chapter is completely self-contained as is its associated appendix, Appendix G, on Grassmann states and the Grassmann oscillator.

We end the thesis in Chapter 7 with a summary and a discussion of prospects for future research.

The original material in this thesis resides in Sections 3.3, 4.2, 4.3, 4.4, 5.2, 5.3, 5.4, 6.2 and 6.3. Each chapter has its own list of references. While this inevitably causes some duplication it has the advantage of making each chapter self-contained. At the end of the thesis there are a number of appendices in which we detail our conventions, state useful formulae and perform calculations whose details are not needed in the text.

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## 2 COSET SPACE DIMENSIONAL REDUCTION

### 2.1 DIMENSIONAL REDUCTION

The idea of all dimensional reduction schemes is to obtain a theory in  $D$  dimensions from one in  $D+d$  dimensions. One possibility is to choose the extra  $d$  dimensions to form a coset space. Then one can expand the fields of the theory in a harmonic expansion. Corresponding to the arbitrary dependence of the fields on the extra coordinates one finds an infinite number of modes. See for example Salam and Strathdee [1].

Other dimensional reduction schemes obtain a finite number of modes by restricting the dependence of the fields on the extra coordinates. One way of doing this is to demand that the Lagrangian in  $D+d$  dimensions be independent of the  $d$  coordinates. In the coset space dimensional reduction scheme one demands that the dependence of the fields on the extra coordinates be a gauge transformation. Then the gauge-invariant Lagrangian is independent of the extra coordinates.

This chapter is essentially a review of the coset space dimensional reduction formalism. As such it contains no original work but does draw from a large number of references [1 - 18]. In the remainder of this chapter references are only given to specific details or models. In the next section we discuss, in general, the application of CSDR to Yang-Mills-Higgs theories. This was the application made by the originators of the scheme, Forgacs and Manton [2]. Commencing with a pure Yang-Mills theory in higher dimensions one obtains after dimensional reduction a Yang-Mills-Higgs theory in four dimensions. We then briefly discuss the inclusion of fermions which is of relevance to Chapter 5. In Section 2.3 we illustrate our discussion with a model by Manton [3]

in which the bosonic sector of the Weinberg-Salam model is derived from a pure Yang-Mills theory in six dimensions. This model is discussed further in Chapter 3 from the point of view of renormalisation.

In Section 2.4 we discuss the extension of the CSDR formalism to the covariant quantisation of theories invariant under Becchi-Rouet-Stora-Tyutin transformations. In this case it is necessary to employ a six-dimensional superspace in which the ordinary coordinates are supplemented by two anticommuting or Grassmann coordinates. We establish our superspace notation and conventions for later use in Chapters 4 and 5 where we perform the BRST quantisation of antisymmetric tensor and spinor-vector gauge theories. In Section 2.5 we illustrate our discussion with the BRST quantisation of a Yang-Mills theory by Delbourgo et al. [16].

## 2.2 YANG-MILLS-HIGGS THEORIES

In this section we rely mainly on the material contained in references [4,5]. For Yang-Mills-Higgs theories the starting point is a pure Yang-Mills theory defined over a space-time which is the direct product of four-dimensional, Minkowski space,  $M^4$ , and a compact coset-space,  $S/R$ . The fields are in a representation of a gauge group  $G$ . The metric is assumed to be block diagonal, respecting the direct product in the space-time, and invariant under  $S$ .

Dimensional reduction from  $4+d$  dimensions to four dimensions occurs by requiring the vector gauge fields,  $A_M$ , to be  $S$ -invariant or  $S$ -symmetric. By this we mean that the action of  $S$  on the gauge fields may be compensated by a gauge transformation. The dependence of the fields,  $A_M$ , on the extra coordinates is then totally fixed but non-trivial. To clarify this we introduce some notation below.

We write our coordinates as

$$x^M = (x^\mu, y^m), \quad \mu=0, \dots, 3, \quad m=5, \dots, 4+d$$

where  $d = \dim(S/R)$ . The Lie algebra of  $S$  is represented by the differential operators  $L_{\xi_{\bar{a}}}$  and the (left) vector fields  $\xi_{\bar{a}}^M$  on the coset space  $S/R$ :

$$[L_{\xi_{\bar{a}}}, L_{\xi_{\bar{b}}}] = C_{\bar{a}\bar{b}}^{\bar{c}} L_{\xi_{\bar{c}}} \quad (2.1)$$

where  $C_{\bar{a}\bar{b}}^{\bar{c}}$  are the structure constants of  $S$ . Under an infinitesimal, left action of  $S$  the coordinates transform as

$$x^M \rightarrow x^M + \rho^M \equiv x^M + \rho^{\bar{a}} \xi_{\bar{a}}^M$$

This induces an action on the vector fields given by

$$L_{\rho} A_M = -\rho^L \partial_L A_M - (\partial_M \rho^L) A_L$$

Then the requirement that the action of  $S$  on  $A_M$  should be compensated by a gauge transformation gives us the following constraint on the fields

$$-\rho^L \partial_L A_M - (\partial_M \rho^L) A_L = \partial_M W^{\rho} - [A_M, W^{\rho}] \equiv D_M(W^{\rho}) \quad (2.2)$$

where  $W^{\rho} = \rho^{\bar{a}} W_{\bar{a}}$ . Applying (2.1) to (2.2) we obtain a condition on the gauge parameter

$$D_M(L_{\rho} W^{\sigma} - L_{\sigma} W^{\rho} + [W^{\rho}, W^{\sigma}] - W^{[\rho, \sigma]}) = 0$$

where  $L_{\rho} W^{\sigma} = -\rho^L \partial_L W^{\sigma}$  and  $[\rho, \sigma]^{\bar{c}} = -\rho^{\bar{a}} \sigma^{\bar{b}} C_{\bar{a}\bar{b}}^{\bar{c}}$ . We solve this in the form of a constraint

$$L_{\rho} W^{\sigma} - L_{\sigma} W^{\rho} + [W^{\rho}, W^{\sigma}] - W^{[\rho, \sigma]} = 0 \quad (2.3)$$

In order to find the  $S$ -invariant fields we first solve the gauge-parameter constraint. We then use these solutions in the field constraint to solve for the fields. The field solutions may be written in terms of several functions,  $\phi$ , of Minkowski-space coordinates only [5]

$$A_{\mu} = A_{\mu}(x^{\mu})$$

$$A_m = \int_{\bar{a}=1}^{\dim S} \phi^{\bar{a}}(x^{\mu}) \xi_{\bar{a}m}^{\bar{c}}(y^m, y^h) |_{y^h=0} \quad (2.4)$$

where  $\bar{\xi}_{am} = \eta_{mn} \bar{\xi}_a^{\bar{n}}$  are the right, vector fields generating  $S$ ,  $\eta_{mn}$  is the metric on  $S$  and the coordinates of  $S$  are split as  $y^{\bar{m}} = (y^m, y^h)$  with  $y^h$  denoting the coordinates of  $R$ .

The  $\phi_a(x^\mu)$  are subject to the constraints

$$[\phi_a, \phi_r] + C_{ar}^{\bar{d}} \bar{\phi}_{\bar{d}} = 0, \bar{a}=1, \dots, \dim S, r=1, \dots, \dim R \quad (2.5)$$

Also the  $A_\mu$  must satisfy

$$[A_\mu, \phi_r] = 0, r=1, \dots, \dim R. \quad (2.6)$$

The  $-\phi_r$  components are constant and generate an  $R$  subalgebra of  $G$ .

In order to have non-trivial solutions of (2.6) we must have an isomorphic image,  $R_G$ , of  $R$  in  $G$ . Then the gauge group preserving the  $S$  symmetry reduces from  $G$  to  $H$ , the centraliser of  $R_G$  in  $G$  i.e.  $H$  is the maximum subgroup of  $G$  whose elements all commute with  $R_G$ . If  $R_G$  contains  $U(1)$  factors then these same factors appear in  $H$ .

In order to dimensionally reduce the theory we substitute our field solutions into a higher dimensional form of the Yang-Mills action

$$S = - \int d^{4+d} x / (-g) F_{MN}^a F_{KL a} \eta^{KM} \eta^{LN} / 4, a=1, \dots, \dim G$$

where the metric is

$$\eta^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta^{mn}(y)/L^2 \end{pmatrix}$$

and  $L$  is an arbitrary length scale for the coset space. We obtain

$$\begin{aligned} S = \Omega \int d^4 x & (-F^{\mu\nu} F_{\mu\nu} / 4 + (D^\mu \bar{\phi}^{\bar{a}})^a (D_\mu \phi_a) / 2L^2 \\ & + (C_{ab}^{\bar{c}} \bar{\phi}_{\bar{c}} + [\phi_a, \phi_b])^a (C_{ab}^{\bar{c}} \bar{\phi}_{\bar{c}} + [\phi_a, \phi_b]) / 4L^4) \end{aligned} \quad (2.7)$$

where  $\Omega$  is the volume of the coset space.

The original Yang-Mills theory in 4+d dimensions has yielded a Yang-Mills-Higgs theory in four dimensions. The  $A_\mu$  remain as gauge fields while the Minkowski-space functions,  $\Phi$ , in the solutions for  $A_m$ , have become the Higgs scalar fields. The original action splits into three parts depending on whether the field strength tensor,  $F_{MN}$ , has neither, one or both indices corresponding to coset-space components. These become the pure gauge term, kinetic term for the Higgs scalars and the Higgs scalar potential respectively.

The Higgs content of (2.7) may be calculated as follows [4]. Suppose the adjoint representation of  $S$ ,  $\text{ad}S$ , decomposes into irreducible representations (irreps) of  $R$  according to the branching rule

$$\text{ad}S \rightarrow \sum_i \underline{r}_i + \text{ad}R$$

where each  $\underline{r}_i$  is an irrep of  $R$ . Similarly, suppose  $\text{ad}G$  decomposes into irreps of  $R_G \times H$  according to the branching rule

$$\text{ad}G \rightarrow \sum_j (\underline{r}'_j \times \underline{h}_j)$$

Then for each pair  $(\underline{r}'_j, \underline{r}_i)$  with  $\underline{r}_i = \underline{r}'_j$  there is a Higgs scalar multiplet  $\underline{h}_j$  in the four dimensional theory.

The original gauge group  $G$  is reduced to  $H$  and the quartic scalar potential leads in general to a spontaneous symmetry breaking of  $H$  to some gauge group  $K$ . Usually one has to explicitly minimise the Higgs potential to find  $K$ . However, in [6] it was shown that if one chooses  $S$  and  $G$  such that there is an isomorphic subgroup,  $S_G$ , of  $S$  in  $G$  then  $K$  is simply the centraliser of  $S_G$  in  $G$ .

To see this assume that  $\phi_{\underline{a}}^a$  takes the value one when  $a = \bar{a}$  and zero otherwise. Assume also that our generators of  $G$  commence with generators of  $S_G$  isomorphic to those of  $S$ . Then these  $\phi_{\underline{a}}^a$  obviously satisfy the constraint (2.4) for all values of  $\bar{a}$  and  $r$ . They are thus

allowed values of Higgs fields. Also, they correspond to the minimum of the scalar potential since the latter vanishes for these values of  $\Phi_a$ . Note this corresponds to zero cosmological constant, at least classically. The unbroken gauge group which leaves the vacuum values of the Higgs fields invariant is then  $K$ , the centraliser of  $S_G$  in  $G$  as stated. Schematically we have

$$G \rightarrow H \rightarrow K \quad \text{with}$$

$$G \supset S_G \times K$$

$$u \quad n$$

$$R_G \times H$$

In summary, if we fix

- i) the groups  $S$  and  $G$
- ii) the group  $R$ , its embedding in  $S$  and the embedding of  $R_G$  in  $G$
- iii) the coupling constant of the higher dimensional theory and
- iv) the size of the coset space  $S/R$

then the four dimensional theory, including the Higgs sector, is completely determined at the classical level.

The above scenario is highly appealing to model builders and in the next section we discuss Manton's derivation of the bosonic sector of the Weinberg-Salam model with a prediction for the weak angle and mass of the Higgs particle [3]. However for realistic model building it is necessary to include fermions.

It is straightforward to add fermions to the CSDR scheme. We discuss this briefly referring the reader in particular to reference [7] for more details. We simply add a Dirac action to the pure Yang-Mills action with the spinors also in some representation of  $G$ . The Dirac action is invariant under local Lorentz transformations as well as gauge transformations. Thus the requirement that the fermionic fields be



S-invariant becomes a constraint on the spinors in which the action of S is compensated by a gauge transformation plus a local Lorentz transformation. This has the form

$$L_\rho \Psi = -\Lambda^\rho \Psi + D(\chi^\rho)$$

where  $\Lambda^\rho = \rho^{\bar{a}} \Lambda_{\bar{a}}$  generates Lorentz transformations and  $D(\chi^\rho)$  gives the gauge transformation. After dimensional reduction the entire theory becomes four-dimensional with the fermions in some multiplets of H. For some examples of the use of CSDR for realistic model building with fermions see [8].

In order to identify the fermion multiplets one first notes that R is naturally embedded in  $SO(d)$  by its action on the tangent space of  $S/R$ . Next one takes the spinor  $s$  of  $SO(d)$  and decomposes it into irreps of R

$$s \rightarrow \sum_i \underline{r}_i$$

Then one decomposes the representation of G,  $g$ , to which the fermions are assigned, into irreps of  $R_G \times H$

$$g \rightarrow \sum_j (\underline{r}'_j \times \underline{h}_j)$$

Then for each pair  $(\underline{r}_i, \underline{r}'_j)$  for which  $\underline{r}_i$  and  $\underline{r}'_j$  are identical irreps there is a multiplet  $\underline{h}_j$  of fermions in the four-dimensional theory.

An interesting application of the CSDR scheme to fermions is that one can achieve parity violation in the four dimensional theory. One way of doing this is to start with Weyl fermions in a complex representation of G provided  $\text{rank } R = \text{rank } S$  [7]. Another way is given in [9]. This is in contrast to Kaluza-Klein type models.

In the next section we give an example of a Yang-Mills-Higgs theory obtained from a pure Yang-Mills theory in higher dimensions to illustrate our discussion.

### 2.3 MANTON'S MODEL

Manton [3] takes his space-time to be  $M^4 \times SO(3)/SO(2)$ . That is the product of four dimensional Minkowski space and a two dimensional sphere of fixed radius  $R$ . The coordinates are  $(x^\mu, \psi, \phi)$  where  $\psi, \phi$  are the polar and azimuthal angles of the sphere. The metric of the space  $\eta^{MN} = \text{Diag}(1, -1, -1, -1, -1/R^2, -1/R^2 \sin^2 \psi)$ . The group  $G$  is unspecified at this stage.

Now the position dependent vectors  $\xi_a^M$  generating  $SO(3)$  transformations are  $\xi_a^\mu = 0$  and

$$\xi_1^\psi = \cos \phi, \xi_2^\psi = \sin \phi, \xi_3^\psi = 0, \xi_1^\phi = -\cot \psi \sin \phi, \xi_2^\phi = -\cot \psi \cos \phi, \xi_3^\phi = 1$$

The requirement that the gauge fields  $A_M$  be  $SO(3)$ -symmetric gives the constraint equation

$$(\partial_M \xi_a^N) A_N + \xi_a^N \partial_N A_M = \partial_M W_a - [A_M, W_a] \quad (2.8)$$

from (2.2) and the constraint equation for  $W_a$  (2.3) becomes

$$\xi_a^K \partial_K W_b - \xi_b^K \partial_K W_a + [W_a, W_b] + C_{ab}^c \bar{c} W_c = 0$$

The solutions to (2.3) may be written as

$$W_1 = \Phi_3 \sin \phi / \sin \psi, W_2 = \Phi_3 \cos \phi / \sin \psi, W_3 = 0 \quad (2.9)$$

where  $\Phi_3$  is a constant element of the Cartan subalgebra (CSA) of  $G$ .

Using these solutions in (2.7) we find the field solutions c.f.(2.4)

$$A_\mu = A_\mu(x), A_\psi = -\Phi_1(x), A_\phi = \Phi_2(x) \sin \psi - \Phi_3 \cos \psi \quad (2.10)$$

where the  $\Phi_i (i=1,2,3)$  are constrained by (c.f.(2.5) and (2.6))

$$\begin{aligned} [\Phi_3, \Phi_1(x)] &= -\Phi_2(x) \\ [\Phi_3, \Phi_2(x)] &= \Phi_1(x) \end{aligned} \quad (2.11)$$

$$[\Phi_3, A_\mu(x)] = 0$$

The standard form of the Yang-Mills density in four dimensions is

$$\int d^4 x -F^{\mu\nu} F_{\mu\nu} / 4g^2. \text{ A natural extension of this to six dimensions is}$$

$$L = \int d^6x (-\det \eta)^{\frac{1}{2}} \kappa (F_{MN}, F_{KL}) \eta^{MK} \eta^{NL} / \bar{g}^2$$

where  $\kappa$  denotes the trace in the adjoint representation and  $\bar{g}$  is the six-dimensional coupling constant. Then one finds on substitution of (2.10) and integration over  $\psi, \phi$

$$L = 4\pi R^2 \int d^4x [\kappa (F_{\mu\nu}, F_{\mu\nu}) - 2\kappa (D_\mu \phi_i, D_\mu \phi_i) / R^2 + \kappa (\epsilon_{ijk} \phi_k + [\phi_i, \phi_j], \epsilon_{ijk} \phi_k + [\phi_i, \phi_j]) / R^4] / \bar{g}^2 \quad (2.12)$$

where  $D_\mu \phi_i = \partial_\mu \phi_i - [A_\mu, \phi_i]$

Since the constraints (2.11) have not yet been applied the gauge group is still  $G$ .

In order to solve the constraints they are rewritten as

$$[\phi_3, \phi] = i\phi$$

$$[\phi_3, \tilde{\phi}] = -i\tilde{\phi} \quad (2.13)$$

$$[\phi_3, A_\mu] = 0 \quad (2.14)$$

where  $\phi = \phi_1 + i\phi_2$  and  $\tilde{\phi} = \phi_1 - i\phi_2$

At this stage Manton constructs a basis for  $G$  based on its root algebra [10]. We note that if  $\tau$  and  $\omega$  are roots then the  $\omega$ -string through  $\tau$  consists of all roots of the form  $\tau + s\omega$  for  $s$  an integer. The commutation relations of  $G$  may be simply described given the roots of  $G$ . First a basis for the Cartan subalgebra is selected

$$[H_i, H_j] = 0 \quad (2.15)$$

Denoting by  $X_\omega$  the generator corresponding to the root  $\omega$  we have

$$[H_i, X_\omega] = \omega_i X_\omega \quad (2.16)$$

$$[X_\omega, X_\omega] = \frac{2\omega_i H_i}{\omega \cdot \omega} = h_\omega$$

$$[X_\omega, X_\tau] = C_{\omega, \tau} X_{\omega+\tau}, \omega \neq -\tau$$

with  $C_{\omega, \tau}$  a constant defined when  $\omega + \tau$  is a root. The basis for  $G$  consists of the  $H_i$  and the  $X_\omega$ .

Consider the commutation relation (2.15).  $\Phi_3$  lies in the CSA of  $G$  so we may write  $\Phi_3 = \Phi_{3i} H_i$ . From (2.15) all members of the CSA of  $G$  commute with  $\Phi_3$ . Now the constraint (2.14) implies that the gauge group after dimensional reduction is the group whose generators all commute with  $\Phi_3$ . We require this group to be  $SU(2) \times U(1)$  which has only two commuting generators hence  $G$  must be a simple group of rank two i.e.  $G = SU(3)$ ,  $O(5)$  or  $G_2$ .

Now consider the relation (2.15). If we choose  $\Phi_3$  orthogonal to some root  $\gamma$  then (2.16) gives

$$[\Phi_3, X_\gamma] = \Phi_{3i} \gamma_i X_\gamma = 0$$

$\Phi_3$  also commutes with  $X_{-\gamma}$  and  $h_\gamma$ ,  $h$  where  $h$  is the element of the CSA orthogonal to  $G$ . Then we can take  $X_\gamma, X_{-\gamma}, h_\gamma, h$  as the generators of  $SU(2) \times U(1)$  or equivalently

$$t_1 = \frac{1}{2}i(X_\gamma + X_{-\gamma}), t_2 = \frac{1}{2}(X_\gamma - X_{-\gamma}), t_3 = \frac{1}{2}ih_\gamma, y = \frac{1}{2}ih$$

Since the  $A_\mu$  belong to  $SU(2) \times U(1)$  we can write

$$A_\mu = A_\mu^1 t_1 + A_\mu^2 t_2 + A_\mu^3 t_3 + B_\mu y$$

Now consider the constraint (2.13).  $\Phi$  has no component in the CSA but it can have a component of  $X_\omega$  if  $\Phi_{3i} \omega_i = i$  for then (2.16) gives

$$[\Phi_3, X_\omega] = \Phi_{3i} \omega_i X_\omega = iX_\omega$$

as required. Now  $\Phi_{3i} \gamma_i = 0$  so we can add any number of  $\gamma$  to  $\omega$  and find that  $\Phi$  has a component of  $X_\tau$  for any  $\tau = \omega + s\gamma$  with  $s$  an integer. If we require that there is one only Higgs scalar doublet after dimensional reduction then the  $\gamma$ -string can only consist of two roots, say  $\alpha$  and  $\beta = \alpha - \gamma$ , so we must choose  $\gamma$  carefully to satisfy this. Then we obtain

$$\Phi = \phi_1 X_\alpha + \phi_2 X_\beta \quad (2.17)$$

Similarly for  $\tilde{\Phi}$  we obtain

$$\tilde{\Phi} = \tilde{\phi}_1 X_{-\alpha} + \tilde{\phi}_2 X_{-\beta} \quad (2.18)$$

Since  $\phi_1$  and  $\phi_2$  were real we have  $\tilde{\phi}_1 = -\phi_1^*$  and  $\tilde{\phi}_2 = -\phi_2^*$ . This requirement of a scalar doublet fixes  $\gamma$  hence  $\phi_3$ . We have finally

$$\phi_3 = \frac{1}{2}i(2 - \frac{1}{2}\langle\gamma, \alpha\rangle)^{-1}(h_\alpha + h_\beta) \quad (2.19)$$

where  $\langle\gamma, \alpha\rangle = 2\gamma \cdot \alpha / (\alpha \cdot \alpha)$

Substituting the solutions (2.17), (2.18), (2.19) to the constraints into (2.12) after some algebra Manton obtains

$$\begin{aligned} L = 4\pi R^2 \int d^4X [ & -(\partial_\mu A_\nu^a + \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c) / (\gamma \cdot \gamma) \\ & - (\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial_\mu B_\nu + \partial_\nu B_\mu) / (\gamma \cdot \gamma) - 2(2Z - 2\phi^\dagger \phi + (\phi^\dagger \phi)^2) / R^4 \alpha \cdot \alpha \\ & + 4(\partial_\mu \phi - \frac{1}{2}i A_\mu^a \sigma^a \phi - \frac{1}{2}i B_\mu \phi \tan \theta)^\dagger (\partial_\mu \phi - \frac{1}{2}i A_\mu^a \sigma^a \phi - \frac{1}{2}i B_\mu \phi \tan \theta) / R^2 \alpha \cdot \alpha ] / \bar{g}^2 \end{aligned}$$

where  $a = 1, 2, 3, \sigma^a$  are the Pauli matrices and  $\phi = (\phi_1, \phi_2)$ .  $\theta$  is the angle between the roots  $\gamma$  and  $\alpha$ .

Rewriting this in canonical form we have

$$\begin{aligned} L = \int d^4X [ & -(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} g A_\mu^b A_\nu^c)(\partial_\mu A_\nu^a + \partial_\nu A_\mu^a + \epsilon^{abc} g A_\mu^b A_\nu^c) / 4 \\ & - (\partial_\mu B_\nu + \partial_\nu B_\mu)(\partial_\mu B_\nu - \partial_\nu B_\mu) / 4 + (\partial_\mu \phi - \frac{1}{2}ig A_\mu^a \sigma^a \phi - \frac{1}{2}ig' B_\mu \phi)^\dagger (\partial_\mu \phi - \frac{1}{2}ig A_\mu^a \sigma^a \phi - \frac{1}{2}ig' B_\mu \phi) \\ & - 16\pi Z / \bar{g}^2 R^2 \alpha \cdot \alpha + \phi^\dagger \phi / R^2 - \frac{1}{2}g^2 (\alpha \cdot \alpha / \gamma \cdot \gamma) (\phi^\dagger \phi)^2 \end{aligned} \quad (2.20)$$

where we have redefined the fields and  $\bar{g}$  by

$$g^2 = \bar{g}^2 \gamma \cdot \gamma / 16\pi R^2$$

$$A_\mu^a = A_\mu^a / g, B_\mu = B_\mu / g, \phi' = (\gamma \cdot \gamma)^{1/2} \phi / g R (\alpha \cdot \alpha)^{1/2}$$

and dropped the primes. Now  $g$  is the SU(2) coupling constant and the U(1) coupling constant is given by

$$g' = g \tan \theta \quad (2.21)$$

Comparing this with  $g' = g \tan \theta_W$  where  $\theta_W$  is the weak angle we have for  $G = SU(3), O(5), G_2$ ,  $\theta_W = 60^\circ, 45^\circ, 30^\circ$  respectively.

The Higgs scalar potential has the form

$$U(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \quad (2.22)$$

where  $\mu^2 = -1/R^2$  and

$$\lambda = g^2/8 \cos^2 \theta. \quad (2.23)$$

If we set  $\phi = v + H$  where  $v^2 = |\mu^2|/2\lambda$  and expand (2.22) about its minimum  $\langle \phi \rangle = v$  then we obtain the quadratic terms.

$$|\mu^2| H^2 + (|\mu^2| g^2/4\lambda) A_\mu^+ A_\mu^- + |\mu^2| A_\mu^Z A_\mu^Z$$

where  $A_\mu^\pm = (A_\mu^1 \pm i A_\mu^2)/\sqrt{2}$ ,  $A_\mu^Z = (g A_\mu^3 - g' B_\mu)/(g^2 + g'^2)^{1/2}$ . Thus we have the

following relationships between the masses of the four-dimensional theory

$$M_H^2 = M_Z^2 \quad (2.24)$$

$$M_W^2 = M_Z^2 \cos^2 \theta \quad (2.25)$$

where  $M_Z$  is the mass of the Z boson,  $M_W$  is the mass of the  $W^\pm$  bosons and the mass of the Higgs is

$$M_H^2 = 2|\mu^2| \quad (2.26)$$

Once the group  $G$  is chosen the only free parameters are  $g$  and  $R$  which may be determined by the electronic charge  $e = g \sin \theta_W$  and by the weak coupling constant,  $G_F$ , related to  $M_W$  by  $M_W^2 = g^2/(32)^{1/2} G_F$ . Then all the parameters of the four dimensional theory are completely determined.

The above model is of interest for two reasons. Firstly, it illustrates how one might solve constraint equations in practice. Secondly, it is an example of a Yang-Mills-Higgs model in which the resultant four dimensional parameters are related to one another through their origin in a higher dimensional pure Yang-Mills theory. See equations (2.21), (2.23), (2.24) and (2.25). It is this aspect

which originally encouraged work on CSDR models. It is hoped that they can reduce the number of arbitrary parameters in grand unified field theories. However the CSDR predictions are only at the classical level. We investigate the effects of renormalisation on the predictive power of this type of model in Chapter 3.

In the next section we examine a different application of the CSDR scheme.

## 2.4 BRST QUANTISATION

The Becchi-Rouet-Stora-Tyutin (BRST) symmetry [11] was first identified as a symmetry of the gauge-fixing plus ghost-compensating Lagrangian in Yang-Mills theories. The BRST transformations mix gauge fields with ghost fields. The BRST symmetry has thus important implications for the quantisation and renormalisation of these theories. Subsequently [12], it was recognised that an extended BRST set could be constructed. This involves a two parameter BRST group where the roles of ghost and antighost are interchanged.

Bonora and Tonin [13] presented a concise derivation of the extended BRST transformation from a superfield formalism. The derivation involves a six-dimensional superspace with coordinates  $(x^\mu, \theta^m)$  with  $m = 5, 6$ . The number of components of the superfield was restricted by imposing zero curvature in the Grassmann directions. Then the BRST transformations simply correspond to translations in  $\theta^m$ . The appropriate space-time symmetry group is inhomogeneous  $O(4) \times T_2$  where the inhomogeneous here refers to ordinary translations.

Delbourgo and Jarvis [14] then extended this space-time symmetry group to a real form of the inhomogeneous supergroup  $OSp(4/2)$ . This group consists of ordinary translations and Lorentz transformations also symplectic transformations in  $\theta^m$ , supertranslations and super

Lorentz transformations. In S-matrix computations the ghost fields cancel with non-physical degrees-of-freedom. Then only physical degrees-of-freedom survive. This is due to the equivalence [15] between tensors of  $Sp(n)$  and formal tensors of  $SO(-n)$ . In graded superspace, Fermi dimensions count negative and the difference between the number of even and odd components of  $OSp(4/2)$ , that is the super-dimension, is equal to the number of degrees-of-freedom of massless gauge fields in Minkowski space-time. This equivalence does not hold for finite spinor representations.

To restrict the number of components of the superfield one can take the space-time to be a product of Minkowski space and a coset-space  $S/R$  then impose the CSDR constraint of S-invariance. In [16] the authors took their space-time to be  $M^4 \times Sp(2) \wedge T_2 / Sp(2)$ . Then they showed that for a Yang-Mills theory with no spontaneous symmetry breaking the S-invariance constraint was equivalent to Bonora and Tonin's flatness condition [13]. We examine this model in Section 2.5. The advantage to using the S-invariance constraint is that it can be easily extended to fermions unlike the flatness condition, although see [17] for an attempt in this direction. The BRST symmetry itself has been elevated beyond Yang-Mills theories to a general principle for constructing gauge theories. Since Chapters 4 and 5 involve BRST quantisation we take this opportunity to establish our superspace notations and conventions.

The space-time supersymmetry appropriate for BRST quantisation is a real form of the inhomogeneous  $OSp(4/2)$  as discussed above. This group preserves the distance

$$(X - Y)^2 = (X - Y)^M \eta_{MN} (X - Y)^N$$



Taking  $X^M = (x^\mu, \theta^m)$  where  $\mu=0,1,2,3$  and  $m=5,6$  we have

$$X^2 = X^M \eta_{MN} X^N = x^\mu x_\mu + \theta^m \theta_m = x^2 + \theta^2$$

The orthosymplectic metric is

$$\eta^{MN} = \begin{pmatrix} \eta^{\mu\nu} & \\ & \eta^{mn} \end{pmatrix}$$

where  $\eta^{\mu\nu}$  is the usual diagonal, Lorentz metric and  $\eta^{mn}$  is the 2x2 antisymmetric matrix

$$\eta^{mn} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The  $\theta^m$  transform as a doublet representation of  $Sp(2)$ . In order to ensure  $X^2$  is real with the usual properties of complex conjugation for anticommuting numbers one cannot take  $\theta^m$  to be real. One possibility is  $\theta_1^* = \theta_1$  and  $\theta_2^* = -\theta_2$  as used by Bonora and Tonin [13]. However the inclusion of the  $Sp(2)$  symmetry ensures that  $\theta_2 = \theta_1^*$  is equally possible. Throughout this thesis we adopt

$$\theta^m = (\theta^5, \theta^6) = (\theta, \bar{\theta})$$

It is useful to introduce the sign factor  $[MN]$  with  $[\mu\nu] = 1 = [\mu\mu]$ ,  $[mn] = -1$  then for example  $X^M X^N = [MN] X^N X^M$ . Note that this implies  $(\theta^5)^2 = (\theta^6)^2 = 0$ . Hence we can expand any superfield  $F(x, \theta)$  around  $\theta = 0$  and obtain at most three terms in the expansion

$$F(x, \theta) = F(x, 0) + \theta^m f_m(x) + \frac{1}{2} \theta^2 f(x)$$

This form of  $\theta$  expansion proves useful in the solution of constraint equations.

In addition to the usual translations and Lorentz transformations of the Poincaré group we have symplectic rotations on  $\theta^m$

$$\theta^m \rightarrow \theta^n \lambda_n^m + \theta^m$$

and supertranslations

$$\theta^m \rightarrow \theta^m + \epsilon^m$$

The superfield actions we construct involve the group invariant measure

$$d^6X = d^4x \, d\bar{\theta} d\theta$$

Note also

$$\int d\theta d\bar{\theta} = 1 = \int d\bar{\theta} d\theta, \quad \int d\theta^m c = 0 \quad (2.27)$$

where  $c$  is a constant.

Now for BRST quantisation we require the action after dimensional reduction to be BRST invariant. Then  $S$  must include supertranslations.

In practice we take

$$S/R = Sp(2) \wedge T_2 / Sp(2)$$

although other choices are possible. Previously, in Section 2.2, we took  $S/R$  to be a compact space since the size of the space appears in the dimensionally reduced action. Here, in view of (2.27), it is permissible for  $S/R$  to be non-compact.

Suppose we write the infinitesimal action of  $S$  on the coordinates as

$$X^M \rightarrow X^M + \rho \bar{a} \xi_a^M = X^M + \rho^M$$

corresponding to the superalgebra  $[J_a^-, J_b^-] = C_{ab}^{\bar{c}} J_c^-$  with

$$[J_{mn}, J_{kl}] = \eta_{mk} J_{nl} + \eta_{nk} J_{ml} + \eta_{ml} J_{kn} + \eta_{nl} J_{km}$$

$$[J_{mn}, P_k] = \eta_{mk} P_n + \eta_{nk} P_m \quad (2.28)$$

$$\{P_k, P_l\} = 0$$

Then the Killing vectors  $\xi_a^M$  are

$$\xi_a^\mu = 0, \quad \xi_k^m = \delta_k^m, \quad \xi_{kl}^m = \theta_k \delta_l^m + \theta_l \delta_k^m \quad (2.29)$$

In the next section we illustrate our discussion and demonstrate the use of our notation with the BRST quantisation of a Yang-Mills theory by Delbourgo et al. [16].

## 2.5 BRST QUANTISATION OF YANG-MILLS THEORY

In reference [16] Delbourgo et al. performed the BRST quantisation of a Yang-Mills theory by coset space dimensional reduction. They took the space-time to be  $M^4 \times Sp(2) \wedge T_2 / Sp(2)$  and the fields in representations of the super-Lorentz group  $OSp(4/2)$  as discussed in the last section.

For the vector gauge fields to be  $Sp(2) \wedge T_2$  invariant the field constraint (2.2) becomes

$$-[\bar{a}K]\xi_a^L \partial_L A_K - (\partial_K \xi_a^L) A_L = \partial_K W_a - ig[A_K, W_a] \quad (2.30)$$

after factorisation of  $\rho^{\bar{a}}$  being careful of sign factors. Similarly the gauge parameter constraint (2.3) becomes

$$\xi_a^K \partial_K W_b - [\bar{a}b]\xi_b^K \partial_K W_a + igW_a W_b - [ab]igW_b W_a = -C_{ab}^{\bar{c}} W_c \quad (2.31)$$

First we examine the gauge parameter constraint. There are three cases to consider (i)  $(\bar{a}, \bar{b}) = (k, l)$ , (ii)  $(kl, mn)$  and (iii)  $(m, kl)$ .

i) The constraint becomes

$$\partial_k W_l + \partial_l W_k - ig\{W_k, W_l\} = 0$$

after substitution of the appropriate structure constants (2.28) and invariant vector fields (2.29). The solution we adopt to this is a pure gauge

$$W_k = i(\partial_k U^{-1})U/g \quad (2.32)$$

since the 'curvature' of  $W_k$  vanishes.  $U$  is given by

$$U = \exp[-ig(\theta_m^m(x) + \frac{1}{2}\theta^2 B(x))] \quad (2.33)$$

where  $\omega_m$  is an anti-commuting Lorentz scalar.

ii) The constraint becomes at  $\theta=0$

$$-ig[W_{kl}, W_{mn}] = \eta_{km} W_{ln} + \eta_{kn} W_{ml} + \eta_{lm} W_{kn} + \eta_{ln} W_{mk}$$

That is  $W_{k1}(x, \theta=0)$  provides an embedding of  $R = Sp(2)$  into  $G$ .

In Section 2.2 we mentioned that non-trivial embeddings of  $R$  into  $G$  lead to symmetry breaking. In this section we only wish to illustrate the quantisation of the Yang-Mills theory so we take  $W_{k1}(x, \theta=0) = 0$ . In reference [18] the authors take

$W_{k1}(x, \theta=0) \neq 0$  and  $G = OSp(n/2)$ . The result is a BRST quantised  $O(n)$  Yang-Mills-Higgs theory in which the curvature in the fermionic directions is non-zero. The Higgs sector of this model is discussed further in Chapter 3.

iii) At  $\theta=0$  the constraint becomes

$$\partial_m W_{k1} = \eta_{km} W_1 + \eta_{1m} W_k \quad (2.34)$$

Now we solve the field constraint (2.30). First we take  $(\bar{a}, k) = (k1, m)$  then the constraint gives at  $\theta=0$

$$-\eta_{km} A_1 - \eta_{1m} A_k = \partial_m W_{k1}$$

By (2.32) and (2.34) this gives

$$A_m(x, \theta=0) = W_m(x, \theta=0) = \omega_m(x) \quad (2.35)$$

Then if we take  $\bar{a}=k$  we obtain from (2.30)

$$-\partial_k A_\mu = \partial_\mu W_k - ig[A_\mu, W_k]$$

$$\partial_k A_m = \partial_m W_k - ig[A_m, W_k] \quad (2.36)$$

Equations (2.32), (2.36) and the boundary condition (2.35) imply that  $A_M(x, \theta)$  is entirely determined from  $U(x, \theta)$  and  $A_\mu(x, \theta=0)$  to be

$$\begin{pmatrix} A_\mu(x, \theta) \\ A_m(x, \theta) \end{pmatrix} = U^{-1} \begin{pmatrix} A_\mu(x, 0) \\ 0 \end{pmatrix} U - i \begin{pmatrix} \partial_\mu \\ \partial_m \end{pmatrix} U^{-1} U/g \quad (2.37)$$

This implies the gauge field strength  $\phi_{MN} = \partial_M \phi_N - [MN] \partial_N \phi_M - \phi_M \phi_N + [MN] \phi_N \phi_M$

is

$$\begin{pmatrix} \phi_{\mu\nu}(x, \theta) \\ \phi_{\mu m}(x, \theta) \\ \phi_{mn}(x, \theta) \end{pmatrix} = U^{-1} \begin{pmatrix} F_{\mu\nu}(x) \\ 0 \\ 0 \end{pmatrix} U$$

where  $F_{\mu\nu}$  is the usual Yang-Mills field strength. Hence the ansatz (2.37) for  $A_M(x, \theta)$  is equivalent to the condition that the curvature in the fermionic directions should vanish.

In reference [14] the authors substituted the ansatz (2.37) into the superaction

$$S = \int d^6x (X^2 \text{str}(\phi^{MN} \phi_{NM}) / 4 + 2A^\mu A_\mu / \mu) \quad (2.38)$$

where  $\text{str}$  denotes the supertrace and  $\mu$  is a real constant. After integration over the coset-space coordinates and appropriate rescalings of  $B$  and  $\omega^m$  (2.38) became

$$S = \int d^4x \left[ -F^{\mu\nu} F_{\mu\nu} / 4 + (\partial^\mu A_\mu B + \frac{1}{2} B^2) - \partial^\mu \omega^m \partial_\mu \omega_m - \frac{1}{2} A^\mu \omega^\nu \partial_\mu \omega_\nu + g^2 \mu (\omega^\mu \omega_\mu)^2 / 8 \right]$$

which is the usual covariant  $\mu$ -gauge Yang-Mills action with modified ghost compensating terms.

In Chapters 4 and 5 we apply this method to the BRST quantisation of antisymmetric-tensor and spinor-vector gauge fields respectively. Before doing so, however, we make a closer examination of the apparent success of CSDR in model building.

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### 3 CSDR AND ONE LOOP CORRECTIONS

#### 3.1 PREAMBLE

The Standard Model in high energy physics leaves many problems unsolved. Amongst these are

- i) the determination of the Higgs sector and
- ii) the existence of chiral fermions.

The CSDR scheme has had some success in addressing both these problems. In this chapter, which is based on the work in reference [1], we are concerned with the apparent success of CSDR models in determining the Higgs sector. With regard to the second problem see reference [2].

In Chapter 2 we discussed in general how one can obtain a Yang-Mills-Higgs theory from a pure Yang-Mills one in higher dimensions using CSDR. The Higgs sector acquires a geometrical significance as the remnant of a higher dimensional gauge potential. The higher dimensional theory has only two free parameters - the overall gauge coupling constant and the size of the coset space. The coupling constants and masses of the four dimensional theory must then be related at the classical level to these two parameters. The Higgs sector is thus completely determined.

In this chapter we address the question of how the relationships between the four dimensional parameters are affected by one loop corrections. If the CSDR models predictions are to hold beyond the classical level then the relationships must hold at one loop order. We obtain the renormalisation constants for a variety of four dimensional models obtained by CSDR and find that the relationships no longer hold beyond the classical level.

Specifically, consider a parameter  $\lambda$ , which could be a quartic scalar coupling or the square of a gauge coupling constant in the



four dimensional theory. Suppose it is related at the classical, unrenormalised level to the overall gauge coupling constant  $g$  by

$$\lambda_u = \alpha g_u^2$$

Then let the one loop renormalised quantities be given by

$$g_u^2 = g_r^2 + g_r^2(\beta g_r^2)/16\pi^2\epsilon$$

$$\lambda_u = \lambda_r + \lambda_r(\gamma g_r^2 + \delta \lambda_r)/16\pi^2\epsilon$$

(in a dimensional regularisation scheme with  $\epsilon=4-d$ ). Then the relationship between renormalised (finite) quantities can only be consistent order by order in  $1/\epsilon$  if

$$\lambda_r = \alpha g_r^2 + O(g_r^4)$$

$$\gamma + \alpha\delta = \beta$$

If these relationships do not hold then the CSDR models have no predictive power beyond the classical level.

Similarly, let  $\mu^2$  be a Higgs scalar mass. After spontaneous symmetry breaking the vector meson masses are given at the classical level by

$$M_u^2 = A\mu_u^2$$

Then let the one loop renormalised quantities be

$$\mu_u^2 = \mu_r^2 + \mu_r^2(B\lambda_r + Cg_r^2)/16\pi^2\epsilon$$

$$M_u^2 = M_r^2 + M_r^2(D\lambda_r + Eg_r^2)/16\pi^2\epsilon$$

Then the implied relationship between renormalised (finite) quantities can only be consistent order by order if

$$M_r^2 = A\mu_r^2 + O(g_r^2)$$

$$B\alpha + D = C\alpha + E$$

Again these relationships must hold if the predictive power of CSDR is to be other than illusory.

In computing renormalisation constants a method powerful enough to handle a wide class of Lagrangians, especially in the scalar sector, is required. For this purpose we adopt the background field method, following for example the expositions of Jack and Osborne [3] of the one loop effective action. Details of this are given in Section 3.2 including an extension to complex scalar representations [4].

In Section 3.3 we study a wide range of models including Manton's model, whose CSDR was discussed in Section 2.3, and the model by Delbourgo and Jarvis whose dimensional reduction was mentioned in Section 2.5 in the context of BRST quantisation. We examine the predictions of the models and whether they can consistently be renormalised to one-loop order. In every case studied we find that the relationships predicted by CSDR cannot be consistently renormalised.

We conclude the chapter in Section 3.4 with a discussion of the models and their renormalisation constants from a higher-dimensional viewpoint.

### 3.2 RENORMALISATION CONSTANTS

In the work of Jack, and Jack and Osborne [3] the heat kernel method, in conjunction with dimensional regularisation, is used to extract infinite parts of functional determinants and various Green functions arising from quantum corrections to the effective action for a theory involving Yang-Mills and scalar fields. In this section we simply quote the resultant formulae, adapted to the Minkowski metric  $(+,-,\dots,-)$  and give the extension of these results to the case of complex scalar fields. Throughout this section and the next our

internal indices may be raised or lowered at will.

The Yang-Mills field is regarded as a Lie-algebra valued vector  $A_\mu = A_\mu^a t_a$ , where  $t_a = -t_a^t$  are real generators in the adjoint representation (with dimension  $d^a$ ) of the gauge group  $G$ . In a more general representation with dimension  $d^\phi$  and generators  $T_a$  (perhaps the representation of the scalar fields) we have  $[T_a, T_b] = f_{abc} T_c$  in terms of the structure constants  $f_{abc} = -(t_a)_{bc}$ . The field strengths and covariant derivatives are defined by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$$

$$D_\mu \phi = (\partial_\mu + A_\mu^a T_a) \phi$$

In a real representation for the scalar fields  $\phi$  we take  $T_a = -T_a^t$ .

In this case it is found that the action

$$S = \int d^4x \left( -F^{\mu\nu a} F_{\mu\nu}^a / 4g^2 + \frac{1}{2} (D^\mu \phi)^t (D_\mu \phi) - U(\phi) \right) \quad (3.1)$$

yields, on quantisation in background Fermi gauges, the one loop infinite effective action

$$\begin{aligned} \Gamma^{(1)}[A_\mu, \phi] = & \int d^4x \left[ (11C/6 - R/12) F^{\mu\nu a} F_{\mu\nu}^a + 4g^2 (D^\mu \phi)^t T^2 (D_\mu \phi) \right. \\ & \left. - (\frac{1}{2} \text{tr}(U'^{\prime\prime})^2 - g^2 (U')^t T^2 \phi + 3 \text{tr}(P^2)/2) \right] / 16\pi^2 \epsilon \end{aligned} \quad (3.2)$$

We have  $d=4-\epsilon$  and the trace information required is

$$\text{tr}(t_a t_b) = -C \delta_{ab} \quad (3.3)$$

$$\text{tr}(T_a T_b) = -R \delta_{ab} \quad (3.4)$$

$$T^2 \phi = \sum_a T_a T_a \phi = (R d^a / d^\phi) \phi \quad (3.5)$$

$$U_{ij} = \partial^2 U / \partial \phi^i \partial \phi^j, \text{tr}(U'^{\prime\prime})^2 = \sum_{ij} U_{ij} U_{ji} \quad (3.6)$$

$$P_{ab} = g^2 (T_a \phi)^t (T_b \phi), \text{tr}(P^2) = \sum_{ab} P_{ab} P_{ba}$$

These one loop infinities may be cancelled if we define the renormalised quantities

$$\begin{aligned} A^u &= Z_A^{\frac{1}{2}} A^r \\ \phi^u &= Z_\phi^{\frac{1}{2}} \phi^r \\ g_u &= Z_A^{\frac{1}{2}} g_r \end{aligned} \quad (3.7)$$

and the appropriate renormalised parameters in the scalar potential  $U(\phi)$  along with the renormalised action (dropping the  $r$  labels)

$$\begin{aligned} S &= \int d^d X \left[ -F^{\mu\nu a} F_{\mu\nu}^a / 4g^2 + \frac{1}{2} (D^\mu \phi)^\dagger (D_\mu \phi) - U(\phi) - (Z_A - 1) F^{\mu\nu a} F_{\mu\nu}^a / 4g^2 \right. \\ &\quad \left. + \frac{1}{2} (Z_\phi - 1) (D^\mu \phi)^\dagger (D_\mu \phi) - \Delta U(\phi) \right] \end{aligned}$$

This leads to the one loop results

$$Z_A^{(1)} = g^2 (22C/3 - R/3) / 16\pi^2 \epsilon \quad (3.8)$$

$$Z_\phi^{(1)} = g^2 (-8T^2) / 16\pi^2 \epsilon \quad (3.9)$$

$$\Delta U_\phi^{(1)} = -(\frac{1}{2} \text{tr}(U'^2) - g^2 (U')^\dagger T^2 \phi + 3 \text{tr}(P^2) / 2) / 16\pi^2 \epsilon \quad (3.10)$$

Note that the introduction of a separate coupling constant renormalisation is avoided because of background gauge invariance [5].

The set of scalar fields  $\phi$  can always be regarded as real, by taking real and imaginary parts for example, but the  $G$  covariance of such a decomposition is not assured. Not all representations can be cast into real form. It is therefore necessary to use an extension of the background field formalism to complex scalar fields  $(\phi, \phi^*)$ . Such an extension has been calculated by Jarvis [4]. We state the results here.

The generators are now taken as anti-hermitian,  $T_a = -T_a^\dagger$ , ensuring the reality of the various traces. Then the gauge invariant action

$$S = \int d^d X \left[ -F^{\mu\nu a} F_{\mu\nu}^a / 4g^2 + (D^\mu \phi)^\dagger (D_\mu \phi) - U(\phi, \phi^*) \right] \quad (3.11)$$

yields, on quantisation in background Fermi gauges, the gauge-invariant, one-loop, infinite, effective action

$$\Gamma^{(1)}(A, \phi, \phi^*) = \int d^d x \left( (11C/6 - R/6) F^{\mu\nu a} F_{\mu\nu}^a + 8g^2 (D^\mu \phi)^\dagger T^2 (D_\mu \phi) - \text{tr}(U'' U''^* + \hat{U}^2) - g^2 (U' T^2 \phi + U'^\dagger T^2 \phi^*) - 3\text{tr}(P+P^T)^2/2 \right) / 16\pi^2 \epsilon \quad (3.12)$$

Apart from obvious factors due to degree-of-freedom doubling, the main complication comes in the one-loop infinite part of the scalar potential.

We have introduced

$$\text{tr}(P+P^T)^2 = \sum_{a,b} (P_{ab} + P_{ba})^2, \quad P_{ab} = g^2 (T_a \phi)^\dagger (T_b \phi)$$

$$U(\phi, \phi^*) = U(0,0) + U' T_\phi + \phi^\dagger U''^* + \frac{1}{2} (\phi^\dagger U \phi + \phi^\dagger U^* \phi^* + \phi^\dagger U'' \phi + \phi^\dagger U''^* \phi^*) \quad (3.13)$$

$R$ ,  $C$  and  $T^2$  are as given in (3.3), (3.4) and (3.5). After introducing the renormalisation constants as in (3.7) and the renormalised action

$$S = \int d^d x \left( -F^{\mu\nu a} F_{\mu\nu}^a / 4g^2 + (D^\mu \phi)^\dagger (D_\mu \phi) - U(\phi, \phi^*) - (Z_A - 1) F^{\mu\nu a} F_{\mu\nu}^a / 4g^2 + (Z_\phi - 1) (D^\mu \phi)^\dagger D_\mu \phi - \Delta U(\phi, \phi^*) \right)$$

we obtain the one-loop results

$$Z_A^{(1)} = g^2 2(11C/3 - R/3) / 16\pi^2 \epsilon$$

$$Z_\phi^{(1)} = g^2 (-8T^2) / 16\pi^2 \epsilon \quad (3.14)$$

$$\Delta U(\phi, \phi^*)^{(1)} = -(\text{tr}(U'' U''^* + \hat{U}^2) - g^2 (U' T^2 \phi + U'^\dagger T^2 \phi^*) + 3\text{tr}(P+P^T)^2/2) / 16\pi^2 \epsilon$$

In the next section we use these results to calculate the one-loop renormalisation constants for a variety of CSDR models.

### 3.3 THE MODELS

In this section we give details of the one loop renormalisation for a wide range of CSDR models. Our models are listed below where our use of the notation  $G$ ,  $H$  and  $M^4 x S/R$  is identical to that in

Chapter 2 and we have  $S_2 \cong SO(3)/SO(2)$  or  $SU(2)/U(1)$  and  $S_3 \cong SO(4)/SO(3)$  or  $SU(2) \times SU(2)/SU(2)$ .

- I. The orthosymplectic supergroup  $G = OSp(n/2)$  over the Grassmann Euclidean space  $M^4 \times Sp(2) \wedge T_2 / Sp(2)$  with  $H = O(n)$  and a real  $\underline{n}$  of scalar Higgs [6];
- II. Rank two groups  $G = SU(3)$ ,  $SO(5)$  or  $G_2$  over  $M^4 \times S_2$  with  $H = SU(2) \times U(1)$  and a complex doublet of Higgs scalars [7].  
The dimensional reduction of this class of models was discussed in Section 2.3;
- III.  $G = SU(n+1)$  over  $M^4 \times S_2$  with  $H = SU(n) \times U(1)$  with a complex  $\underline{n}$  of Higgs scalars [8] and
- IV.  $G = SU(r(2l+1))$  over  $M^4 \times S_3$  with  $H = SU(r)$  and an adjoint plus singlet, real  $(r^2 - 1 + 1)$ , of Higgs scalars [8].

Case I is of interest since it originates from a gauged supergroup over a super-coset space. It was mentioned briefly in connection with BRST quantisation in Section 2.5.

Case II, and its generalisation for unitary groups case III, provide illustrations of one loop corrections for  $H$  non-simple. That is there are at least two different gauge couplings. In case IV  $H$  is simple but the Higgs sector is more complex. The two scalar multiplets give rise to five different quartic couplings each receiving separate renormalisation. Included in case IV are models where the spontaneous breaking of  $H$  to  $K$  is geometrically determined. For example  $r=5$ ,  $l=\frac{1}{2}$  has been discussed in this connection [9].

We now discuss these models in detail.

#### CASE I

In this model [6] we have

$G = OSp(n/2)$  over  $M^4 \times Sp(2) \wedge T_2 / Sp(2)$  with  $H=O(n)$  and a real  $\underline{n}$  of

Higgs scalars. The  $O(n)$  generators are  $T_{ij} = -T_{ji}$ ,  $i=1, \dots, n$  and satisfy the commutation relations

$$[T_{ij}, T_{kl}] = \delta_{jk} T_{il} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik} + \delta_{il} T_{jk} = f_{ijkl}^{mn} T_{mn} \quad (3.15)$$

In the vector representation we have for example

$$(T_{ij})_m^n = \delta_j^n \delta_{im} - \delta_i^n \delta_{jm} \quad (3.16)$$

To calculate the renormalisation constants (3.7) we need to find  $R$ ,  $C$  and  $T^2$ . Traces are normalised to  $\delta_{ij,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})$ . We have from (3.16)

$$\begin{aligned} \text{Tr}(T_{ij} T_{kl}) &= (\delta_j^n \delta_{im} - \delta_i^n \delta_{jm})(\delta_l^m \delta_{kn} - \delta_k^m \delta_{ln}) \\ &= -4\delta_{ij,kl} \end{aligned}$$

From (3.4) this implies  $R=4$ . In the adjoint representation we have generators  $t_{ij}$  with  $(t_{ij})_{kl,mn} = f_{ijklmn}$ . From (3.15) this implies

$$\begin{aligned} \text{tr}(t_{ij} t_{kl}) &= f_{ij}^{pqmn} f_{klmnpq} \\ &= -4(n-2)\delta_{ij,kl} \end{aligned}$$

Hence  $C = 4(n-2)$  by (3.3). Finally for  $T^2$  we have

$$\begin{aligned} (T^2)_z^x &= \frac{1}{2}(T^{ij})_z^y (T_{ij})_y^x \\ &= -(n-1)\delta_z^x \end{aligned}$$

from (3.16). Hence  $T^2 = -(n-1) \cdot 1$  from (3.5).

Now the model is defined by the action [5]

$$S = \int d^d x (-F^{\mu\nu ij} F_{\mu\nu ij} / 8g^2 + \frac{1}{2}(D_\mu \phi^i)^2 - 3(\phi^i \phi_i)^2 g^2 / 8)$$

This is a model with  $n$  massless scalars and a quartic coupling given at the classical level by

$$\lambda = 3g^2/8. \quad (3.17)$$

From (3.8), (3.9) and the values of  $R, C, T^2$  we find

$$Z_A^{(1)} = (88n/3 - 60)g^2/16\pi^2\epsilon \quad (3.18)$$

$$Z_\phi^{(1)} = 8(n-1)g^2/16\pi^2\epsilon \quad (3.19)$$

To find the one loop infinite correction to the scalar potential we have  $U(\phi) = -\lambda\phi^4$ ,  $U_i = -4\lambda\phi^2\phi_i$ ,  $U_{ji} = -8\lambda\phi_j\phi_i - 4\lambda\phi^2\delta_{ji}$  hence

$$\text{tr}(U^{ii})^2 = (16n+128)\lambda^2(\phi^i\phi_i)^2. \text{ Also from (3.6)}$$

$$\begin{aligned} \text{Tr}(P^2) &= \sum_{a,b=1}^n g^4 (T^a)^{ik} \phi_k (T^b)_i^l \phi_l (T_b)^{jm} \phi_m (T_a)_j^n \phi_n \\ &= \frac{1}{2} g^4 (\delta^{gi} \phi^h - \delta^{hi} \phi^g) (\delta_i^f \phi^d - \delta_i^d \phi^f) (\delta_d^j \phi^f - \delta_f^j \phi_d) (\delta_{hj} \phi_g - \delta_{gj} \phi_h) \\ &= g^4 (n-1) \phi^4 \end{aligned}$$

Hence (3.10) gives

$$\Delta U(\phi)^{(1)} = (3(n-1)g^4/2 + 4(n-1)\lambda g^2 + 8(n+8)\lambda^2)(\phi^i\phi_i)^2/16\pi^2\epsilon$$

Introducing, in addition to (3.7),  $\lambda_u = Z_\lambda Z_\phi^{-2} \lambda_r$  we have

$$\Delta U(\phi)^{(1)} = -\lambda Z_\lambda^{(1)} (\phi^i\phi_i)^2 \text{ hence}$$

$$Z_\lambda^{(1)} = ((8n+8)\lambda + 4(n-1)g^2 + 3(n-1)g^4/2\lambda)/16\pi^2\epsilon \quad (3.20)$$

Note that this result, for  $n=2$ , is in agreement with the  $\alpha=1$  result of Fukuda and Kugo [10].

Now the relationship (3.17) which holds at the classical level implies  $Z_\lambda Z_\phi^{-2} \lambda_r = 3Z_A^{-1} g_r^2/8$ . If (3.17) is to hold under renormalisation then we must have  $\lambda_r = 3g_r^2/8$  hence  $Z_\lambda Z_\phi^{-2} = Z_A^{-1}$ . At the one loop level this implies

$$Z_\lambda^{(1)} - 2Z_\phi^{(1)} + Z_A^{(1)} = 0 \quad (3.21)$$



However from (3.18), (3.19) and (3.20) we find

$$73n/3 - 28 \neq 0$$

after substitution for  $g^r$  and  $\lambda^r$ . Thus the classical prediction of this CSDR model has failed to survive renormalisation.

In this model the CSDR relation  $\mu^2 = 0$  is consistent at first order in that no mass counter terms is generated. That quantum corrections may drive the spontaneous symmetry breaking is an interesting possibility. However a straightforward application of the Coleman-Weinberg formalism [11] requires  $\lambda$  of  $O(g^2)$  which never arises in CSDR schemes.

$$H = SU(n) \times U(1)$$

Notation for cases II and III may be introduced together as both refer to a  $SU(n) \times U(1)$  theory. Note that both cases have a complex multiplet of Higgs. Comprehensive tabulations of Casimir eigenvalues and trace invariants for  $SU(n)$  are available in reference [12].

The  $SU(n)$  generators  $T_a$  with  $a=1, \dots, n^2-1$  in the fundamental representation  $\{1\}$  are  $T_a = -\frac{1}{2}i\lambda_a$  where  $\lambda_a$  is a standard trace-normalised Gell-Mann matrix,  $\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$ . It is convenient to fix the additional  $U(1)$  generator relative to  $\lambda_0 = (2/n)^{\frac{1}{2}}.1$  which has the same normalisation as the  $\lambda_a$ . We take  $T_0 = -\frac{1}{2}i\xi\lambda_0$ . Then from (3.3) and (3.5) for the scalar fields (subscripts distinguish between  $SU(n)$  and  $U(1)$ ) we have

$$R_n = \frac{1}{2}, R_1 = \frac{1}{2}\xi^2$$

$$T_n^2 = -((n^2-1)/2n).1, T_0^2 = -(\xi^2/2n).1$$

since the dimension of the adjoint representation is  $(n^2-1)$ . Also for the adjoint representation we have from (3.4)

$$C_n = n, C_1 = 0$$

In the formulae (3.11) to (3.14) the distinction between the  $SU(n)$  and  $U(1)$  factors arises by regarding  $g$  (which always appears in association with the generators  $T$ ) as a diagonal matrix with  $g_{ab} = g_n \delta_{ab}$  for  $T_a, T_b \in SU(n)$  and  $g_{00} = g_1 \delta_{00}$  for  $T_0 \in U(1)$ .

With these preliminaries we quickly obtain for a  $SU(n) \times U(1)$  model with complex scalars in the  $\{1\} \times \{\xi\}$  representation

$$\begin{aligned} Z_n^{(1)} &= (22n/3 - 1/3)g_n^2/16\pi^2\epsilon \\ Z_1^{(1)} &= (-\xi^2/3)g_1^2/16\pi^2\epsilon \end{aligned} \quad (3.22)$$

from (3.14) and

$$Z_\phi^{(1)} = (4(n^2-1)g_n^2/n + 4\xi^2 g_1^2/n)/16\pi^2\epsilon$$

Now if we write the scalar potential in the form

$$U(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

and define renormalised parameters by

$$|\mu_u^2| = Z_\mu Z_\phi^{-1} |\mu_r^2|, \quad \lambda_u = Z_\lambda Z_\phi^{-2} \lambda_r \quad (3.23)$$

then the renormalised scalar potential is

$$U(\phi) = -|\mu^2| Z_\mu \phi^\dagger \phi + \lambda Z_\lambda (\phi^\dagger \phi)^2. \quad (3.24)$$

From (3.14) we find the one-loop infinite correction to the scalar potential is

$$\begin{aligned} -U^{(1)}(\phi) &= \mu^2 ((n^2-1)g_n^2/n + \xi g_1^2/n + 4(n+1)\lambda) \phi^\dagger \phi / 16\pi^2\epsilon \\ &+ (4(n+4)\lambda^2 + 2(n^2-1)\lambda g_n^2/n + 2\xi \lambda g_1^2/n + 3(n-1)(n^2+2n-2)g_n^4/4n^2 \\ &+ 3(n-1)\xi^2 g_1^2 g_n^2/n^2 + 3\xi^4 g_1^4/2n^2) (\phi^\dagger \phi)^2 / 16\pi^2\epsilon \end{aligned} \quad (3.25)$$

Using (3.24) we can then easily read off  $Z_\mu^{(1)}$  and  $Z_\lambda^{(1)}$ . The reader who wishes to omit the detail behind (3.25) should go straight to our second model.

To obtain (3.25) from (3.14) we need to calculate the following terms (i)  $(P_{pq} + P_{qp})^2$ , (ii)  $\text{tr}(U'^2 + \hat{U}^2)$  and (iii)  $(U'^t(gT)^2\phi + U'^t(gT)^2\phi^*)$

i) We have  $p \in (0, a)$  with  $a = 1, \dots, n^2 - 1$  and  $(P_{pq} + P_{qp})^2 = 2(P_{pq}P_{pq} + P_{pq}P_{qp})$

$$\text{Now } P_{pq} = ((gT)_p\phi)^\dagger ((gT)_q\phi) = -\phi^\dagger (gT)_p (gT)_q \phi \text{ by } T_p^\dagger = -T_p$$

$$\text{Interchanging } T_p \text{ and } T_q \text{ this becomes } P_{pq} = P_{qp} - g_{pr}g_{qs}f_{rst}\phi^\dagger T_t\phi$$

The second term is antisymmetric in  $b$  and  $a$  so multiplying by

$P_{pq}$  we have

$$\begin{aligned} P_{pq}P_{pq} &= P_{pq}P_{qp} - \frac{1}{2}(P_{pq} - P_{qp})g_{pr}g_{qs}f_{rst}\phi^\dagger T_t\phi \\ &= P_{pq}P_{qp} + \frac{1}{2}\tilde{f}_{pqs}\tilde{f}_{pqt}\phi^\dagger T_s\phi\phi^\dagger T_t\phi \end{aligned}$$

where  $\tilde{f}_{pqs} = g_{pr}g_{qt}f_{rts}$ . Now  $\tilde{f}_{pqr}$  is only non-zero when all its indices are in  $SU(n)$ . Then  $\tilde{f}_{abc} = g_n^2 f_{abc}$ . Also  $f_{abc} = -(t_c)_{ab}$   
 $\tilde{f}_{pqs} T_s \tilde{f}_{pqt} T_t = g_n^4 (t_a)_{bc} (t_d)_{bc} T_a T_d = C_n g_n^4 T_a T_a$ .

Also  $P_{pq}P_{qp} = P_{ab}P_{ba} + 2P_{ao}P_{oa} + P_{oo}P_{oo}$ . Hence

$$\begin{aligned} P_{pq}P_{pq} + P_{pq}P_{qp} &= 2g_n^4 \phi^\dagger T_a T_b \phi \phi^\dagger T_b T_a \phi + 4g_1^2 g_n^2 \phi^\dagger T_a T_b \phi \phi^\dagger T_b T_a \phi \\ &+ 2g_1^4 \phi^\dagger T_o^2 \phi \phi^\dagger T_o^2 \phi + \frac{1}{2}g_n^4 C_n \phi^\dagger T_a \phi \phi^\dagger T_a \phi \end{aligned}$$

$$\text{Now } T_o \phi^\dagger \phi T_o = -\xi^2 \phi \phi^\dagger / 2n \text{ and}$$

$$\begin{aligned} (T_a \phi \phi^\dagger T_a)_\alpha^\beta &= -(\lambda_a)_\alpha^\gamma \phi_\gamma (\phi^\dagger)^\rho (\lambda_a)_\rho^\beta \\ &= -\frac{1}{2}(\phi^\dagger \phi) \delta_\alpha^\beta + \phi_\alpha (\phi^\dagger)^\beta / 2n \end{aligned}$$

where we have used the completeness relation

$$(\lambda_a)_\alpha^\beta (\lambda_a)_\gamma^\rho = 2\delta_\gamma^\beta \delta_\alpha^\rho - 2\delta_\alpha^\beta \delta_\gamma^\rho / n \quad \alpha = 1, \dots, n$$

Using these to evaluate our expression for  $(P_{pq} + P_{qp})^2$  and  $C_n = n$ ,

we obtain

$$(P_{pq} + P_{qp})^2 = 2[g_n^4(n-1)(n^2+2n-2)/4n^2 + \xi^2 g_1^2 g_n^2(n-1)/n^2 + g_1^4 \xi^4 / 2n^2](\phi^\dagger \phi)^2$$

ii) From (3.13) and  $U(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$  we obtain

$$U_{\alpha\beta} = 2\lambda \phi_\alpha \phi_\beta^\dagger + 2\lambda \delta_{\alpha\beta} \phi^\dagger \phi + \mu^2 \delta_{\alpha\beta}, \quad \hat{U}_{\alpha\beta} = 2\lambda \phi_\alpha^* \phi_\beta^\dagger$$

hence

$$\begin{aligned} \text{tr}(U'^2) &= \text{tr}(2\lambda \phi_\alpha \phi_\beta^\dagger + 2\lambda \delta_{\alpha\beta} \phi^\dagger \phi + \mu^2 \delta_{\alpha\beta})(2\lambda \phi_\alpha \phi_\beta^\dagger + 2\lambda \delta_{\alpha\beta} \phi^\dagger \phi + \mu^2 \delta_{\alpha\beta}) \\ &= n\mu^4 + 4\lambda\mu^2(n+1)\phi^\dagger \phi + 4\lambda^2(n+3)(\phi^\dagger \phi)^2 \end{aligned}$$

$$\text{tr}(\hat{U})^2 = 4\lambda^2(\phi^\dagger \phi)^2$$

iii) We have  $(U'(gT)^2 \phi + U'^\dagger(gT)^2 \phi^*) = 2U'(gT)^2 \phi$  and

$$U' = \mu^2 \phi^\dagger \phi + 2\lambda \phi^\dagger \phi \phi^\dagger. \quad \text{Now } -(gT)^2 = g_n^2(n^2-1)/n + g_1^2 \xi^2/n \text{ hence}$$

$$\begin{aligned} U'^T(gT)^2 \phi + U'^\dagger(gT)^2 \phi^* &= -\mu^2((n^2-1)g_n^2/n + \xi^2 g_1^2/n) \phi^\dagger \phi \\ &\quad - 2\lambda((n^2-1)g_n^2/n + \xi^2 g_1^2/n)(\phi^\dagger \phi)^2 \end{aligned}$$

From (i), (ii) and (iii) we obtain (3.25) as stated.

## CASE II

We now specialise to Manton's model [7] of the bosonic sector of the standard electroweak theory. The dimensional reduction of this model was discussed in Section 2.3. We have  $G = SU(3)$ ,  $SO(5)$  or  $G_2$  over  $M^4 \times S_2$  with  $H = SU(2) \times U(1)$  and a complex doublet of Higgs scalars. For this  $H$  we have  $n=2$  in our  $SU(n) \times U(1)$  formulae. In order to make contact with our notation consider the form of the covariant derivative given in (2.20). There

$$D_\mu \phi = \partial_\mu \phi + g A_\mu^a (-\frac{1}{2} i \sigma_a) \phi + g (-\frac{1}{2} i 1) B_\mu \tan \theta$$

where  $g$  is the overall coupling constant and  $\theta = 60^\circ, 45^\circ, 30^\circ$  for  $G = SU(3), SO(5)$  or  $G_2$  respectively. In our notation we have

$$D_\mu \phi = \partial_\mu \phi + g_2 A_\mu^a (-\frac{1}{2} i \lambda_a) \phi + g_1 A_\mu^0 (-\frac{1}{2} i \xi (2/n)^{\frac{1}{2}} 1) \phi$$

Hence  $\xi=1$  and

$$g_1 = g_2 \tan \theta \quad (3.26)$$

In Section 2.3 we found the following relationships between parameters, see equations (2.21), (2.23), (2.24) and (2.25)

$$\lambda = g_2^2 / 8 \cos^2 \theta \quad (3.27)$$

$$\mu^2 = -1/R^2$$

where  $R$  is the radius of  $S_2$  and

$$M_H^2 = M_Z^2 \quad (3.28)$$

$$M_W^2 = M_Z^2 \cos^2 \theta \quad (3.29)$$

where  $M_H$  is the Higgs meson mass

$$M_H^2 = 2|\mu^2| \quad (3.30)$$

and the  $Z$  and  $W^\pm$  boson masses are

$$M_Z^2 = 2|\mu^2| \quad (3.31)$$

$$M_W^2 = 2|\mu^2| \cos^2 \theta \quad (3.32)$$

Equations (3.26), (3.27), (3.28) and (3.29) are the predictions of this CSDR model at the classical level. It is these relationships whose validity or otherwise must be checked under renormalisation.

From (3.22) to (3.25) we have (for  $n=2$ ,  $\xi=1$ )

$$Z_2^{(1)} = (43g_2^2/3)/16\pi^2\epsilon$$

$$Z_1^{(1)} = (-g_1^2/3)/16\pi^2\epsilon$$

$$Z_\phi^{(1)} = (6g_2^2 + 2g_1^2)/16\pi^2\epsilon \quad (3.33)$$

$$Z_\mu^{(1)} = (3g_2^2/2 + g_1^2 + 12\lambda)/16\pi^2\epsilon$$

$$Z_\lambda^{(1)} = (24\lambda + 3g_2^2 + g_1^2 + 9g_2^4/8\lambda + 3g_1^2g_2^2/4\lambda + 3g_1^4/8\lambda)/16\pi^2\epsilon$$

Consider first the relationship (3.26). This implies by (3.7)

$$Z_1^{-1/2} g_1^r = Z_2^{-1/2} g_2^r \tan \theta$$

Now if the relationship is to hold under renormalisation then we must have  $g_1^r = g_2^r \tan \theta$ . Hence

$$-1/2 Z_1^{(1)} = -1/2 Z_2^{(1)} \quad (3.34)$$

must hold. However from (3.33) we find (substituting for  $g_1^r$  in terms of  $g_2^r$ )

$$Z_2^{(1)} - Z_1^{(1)} = (43 + \tan^2 \theta) g_2^r{}^2 / 3 \neq 0 \quad (3.35)$$

Similarly if (3.27) is to hold under renormalisation then we must have  $Z_\lambda Z_\phi^{-2} = Z_2^{-1}$  or to one loop order

$$Z_\lambda^{(1)} - 2Z_\phi^{(1)} + Z_2^{(1)} = 0 \quad (3.35)$$

However from (3.33) after substitution for  $g_1^r$  and  $\lambda^r$  we obtain

$$g_2^r{}^2 (18 \cos^4 \theta + 25 \cos^2 \theta + 9) / 3 \cos^3 \theta \neq 0 \quad (3.37)$$

It is the failure of (3.34) and (3.36) which indicate the relationships predicted by CSDR between gauge coupling constants do not survive one loop renormalisation.

The mass relationships (3.28) and (3.29) must be checked by identifying the renormalised masses after spontaneous symmetry breaking.

The scalar sector of our theory is

$$\begin{aligned} L(\phi) = & -|\mu|^2 |Z_\mu \phi^\dagger \phi - \lambda Z_\lambda (\phi^\dagger \phi)^2 \\ & + Z_\phi (\partial_\mu \phi + g_2 A_\mu^a (-1/2 i \lambda_a) \phi + g_1 A_\mu^0 (-1/2 i.1) \phi)^\dagger \\ & \times (\partial_\mu \phi + g_2 A_\mu^a (-1/2 i \lambda_a) \phi + g_1 A_\mu^0 (-1/2 i.1) \phi) \end{aligned} \quad (3.38)$$

If we set  $\phi = v + H$  where  $v = (|\mu|^2 |Z_\mu / 2\lambda Z_\lambda|)^{1/2}$  and expand (3.38) about the minimum  $\langle \phi \rangle = v$  then we obtain the quadratic terms

$$\frac{1}{2}(2|\mu^2|Z_\mu)H^2 + (g_2^2|\mu^2|Z_\mu Z_\phi/4\lambda Z_\lambda)(A_\mu^+ A_\mu^-) + \frac{1}{2}((g_1^2 + g_2^2)|\mu^2|Z_\mu Z_\phi/4\lambda Z_\lambda)A_\mu^2 A_\mu^2 \quad (3.39)$$

where  $A_\mu^\pm = (A_\mu^1 \pm iA_\mu^2)/\sqrt{2}$ ,  $A_\mu^2 = (g_2 A_\mu^3 - g_1 A_\mu^0)/(g_1^2 + g_2^2)^{1/2}$ . The coefficients of these terms give the renormalised masses of the Higgs meson H, the  $W^\pm$  and Z bosons respectively.

Using  $M_H^{r2}$ ,  $M_W^{r2}$  and  $M_Z^{r2}$  from (3.39) and (3.30), (3.31), (3.32) for the unrenormalised masses allows us to write

$$\begin{aligned} M_H^{u2} &= M_H^{r2} Z_\mu^{-1} \\ M_W^{u2} &= M_W^{r2} (8\lambda \cos^2 \theta / g_2^2) Z_\lambda Z_\mu^{-1} Z_\phi^{-1} \\ M_Z^{u2} &= M_Z^{r2} (8\lambda / (g_1^2 + g_2^2)) Z_\lambda Z_\mu^{-1} Z_\phi^{-1} \end{aligned} \quad (3.40)$$

Now the relationships (3.28) and (3.29) imply

$$M_H^{r2} Z_\mu^{-1} = M_Z^{r2} (8\lambda / (g_1^2 + g_2^2)) Z_\lambda Z_\mu^{-1} Z_\phi^{-1} \quad (3.41)$$

$$M_W^{r2} (8\lambda \cos^2 \theta / g_2^2) Z_\lambda Z_\mu^{-1} Z_\phi^{-1} = M_Z^{r2} (8\cos^2 \theta \lambda / (g_1^2 + g_2^2)) Z_\lambda Z_\mu^{-1} Z_\phi^{-1} \quad (3.42)$$

If (3.29) is to hold also for renormalised masses and parameters then (3.39) implies we must have

$$0 = Z_\lambda^{(1)} - Z_\phi^{(1)}$$

to one loop order. Note that we are only concerned with infinite contributions to (3.41) here. However using (3.33) for  $Z_\lambda^{(1)}$  and  $Z_\phi^{(1)}$ , and the expressions (3.26), (3.27) for the renormalised couplings we obtain

$$Z_\lambda^{(1)} - Z_\phi^{(1)} = g_2^2 (6\cos^4 \theta - 2\cos^2 \theta + 5) / \cos^2 \theta \neq 0$$

Hence the prediction  $M_H = M_Z$  cannot be consistently renormalised. The relationship (3.42) implies  $M_W^{r2} = M_Z^{r2} g_2^2 / (g_1^2 + g_2^2)$ . However this ceases to be a prediction of CSDR for the W mass since the coupling constant relationships assumed in its evaluation are inconsistent at one loop.

We now examine some other CSDR models and their predictions. For brevity we only consider the coupling constant predictions in detail sketching briefly the spontaneous symmetry breaking which is of phenomenological interest only in the  $SU(2) \times U(1)$  case.

### CASE III

In this model we have  $G = SU(n+1)$  over  $M^4 \times S_2$  with  $H = SU(n) \times U(1)$  and a complex  $n$  of Higgs scalars [8].

The model is a direct generalisation of the choice  $G = SU(3)$  with  $\theta = 60^\circ$  in case II. The notation and renormalisation constants (3.22) for a model with  $H = SU(n) \times U(1)$  have already been given. Here we simply state the relationships predicted at the classical level by CSDR and examine their one loop corrections.

From Schwarz and Tyupkin [7] equations (21), (26), (35), (36) and in particular for  $n=2$  equation (37) we again obtain  $\xi=1$  and we have

$$g_1 = g_n \sqrt{n+1} \quad (3.43)$$

$$\lambda = \frac{1}{2} g_n^2 \quad (3.44)$$

after expressing the action in the form (3.11). These agree for  $n=2$  with the  $SU(3)$  version ( $\theta=60^\circ$ ) of (3.26) and (3.27). Once more  $\mu^2 = -1/R^2$  where  $R$  is the radius of  $S_2$ . Spontaneous symmetry breaking to  $K = SU(n) \times U(1)$  occurs with a singlet 'Higgs' meson and a complex  $n-1$  'W' plus singlet 'Z' vector bosons with predicted mass relationships analogous to (3.28) and (3.29) which we do not study in detail.

For the coupling constant renormalisations we use  $Z_n^{(1)}$ ,  $Z_1^{(1)}$  and  $Z_\phi^{(1)}$  from (3.22) and  $Z_\lambda^{(1)}$  from (3.25) with  $\xi=1$ . The discussion leading to equations (3.34) and (3.36) is identical with  $Z_n^{(1)}$  replacing  $Z_2^{(1)}$ . After substitution for  $\lambda^r$  and  $g_1^r$  with the appropriate



renormalised versions of (3.43) and (3.44) we find

$$Z_n^{(1)} - Z_1^{(1)} = 23ng_n^2/3 \neq 0 \quad (3.45)$$

$$Z_\lambda^{(1)} - 2Z_\phi^{(1)} + Z_n^{(1)} = (29n+73)g_n^2/6 \neq 0 \quad (3.44)$$

Note these agree with (3.35) and (3.37) for  $n=2$  and  $\theta=60^\circ$  as they must. Once again the predictions of CSDR at the classical level do not survive renormalisation.

#### CASE IV

In this model [8] we have  $G = SU(r(2l+1))$  over  $M^4 \times S_3$  with  $H = SU(r)$  and an adjoint plus singlet, real  $(\underline{r^2-1} + \underline{1})$  of Higgs scalars.

It is convenient for this model to adopt a vector notation for fields in the adjoint representation. Since  $T^2 = -r.1$ ,  $C = r$ ,  $R = \frac{1}{2}$  (see case II) and all traces vanish for the singlet representation we have from (3.8) and (3.9).

$$Z_A^{(1)} = 7rg^2/16\pi^2\epsilon$$

$$Z_\phi^{(1)} = 8rg^2/16\pi^2\epsilon$$

$$Z_{\phi_0}^{(1)} = 0 \quad (3.47)$$

From Schwarz and Tyupkin [8] three equations after (38) we find

$$U(\phi) = g^2 \text{tr}(\phi^2)^2/4l(l+1) \quad (3.48)$$

after appropriate redefinitions of fields to obtain the action in the form (3.1).  $R$  is the radius of  $S_3$  and  $\phi$  is the matrix  $\phi = \underline{\phi} \cdot \underline{\lambda} + \phi_0 \lambda_0$  where  $\lambda_a, \lambda_0$  were given in case I. In order to compute the one loop corrections to this we need to expand it in the form

$$U(\phi) = +\lambda \phi_0^4 + \lambda_s (\underline{\phi}^2)^2 + \lambda_m \underline{\phi}^2 \phi_0^2 + \lambda_d \phi_0 \underline{\phi} \cdot \underline{\phi} + \lambda_{dd} (\underline{\phi} \star \underline{\phi})^2 \quad (3.49)$$

where  $(\underline{\phi} \star \underline{\phi})_a = d_{abc} \phi_b \phi_c$ . There are five possible quartic invariants

involved. The scalar fields may be regarded as constituting the reducible representation  $\{\bar{1}\}\{1\}$  of  $U(r)$ . The symmetric fourth Kronecker power is  $\{\bar{1}\}\{1\} \otimes \{4\} = \sum_{|\lambda|=4} \{\bar{\lambda}\}\{\lambda\}$ . Each of the five partitions of 4 produces just one quartic invariant. To evaluate the trace in (3.48) we use some formulae for the calculus of  $SU(r)$  invariant tensors given in Appendix B. First we write  $\Phi = \phi_a \lambda_a + \phi_0 \lambda_0$ . Since  $\lambda_0$  commutes with  $\lambda_a$  we find

$$\begin{aligned} \Phi^4 = & \phi_0^4 \lambda_0^4 + 4\phi_0^3 \phi_a \lambda_0^3 + 6\phi_0^2 \phi_a \phi_b \lambda_0^2 \lambda_a \lambda_b + 4\phi_0 \phi_a \phi_b \phi_c \lambda_0 \lambda_a \lambda_b \lambda_c \\ & + \phi_a \phi_b \phi_c \phi_d \lambda_a \lambda_b \lambda_c \lambda_d \end{aligned}$$

Using (B.1), (B.2), (B.7) and  $\lambda_0 = (2/r)^{1/2}$  we obtain

$$\text{tr}(\Phi^4) = 4\phi_0^4/r + 4(\phi_a^2)^2/r + 24\phi_0^2 \phi_a^2/r + 8(2/r)^{1/2} \phi_0 \phi_a \phi_b \phi_c + 2(\phi_a \phi_b)^2$$

where  $(\phi_a \phi_b)_a = d_{abc} \phi_b \phi_c$ . Substituting this into (3.48) and comparing with (3.49) yields

$$\lambda = \lambda_s = \lambda_m/6 = g^2/r1(1+1)$$

$$\lambda_d = 2\sqrt{2}g^2/\sqrt{r}1(1+1) \quad (3.50)$$

$$\lambda_{dd} = g^2/21(1+1)$$

These are the relationships, predicted by CSDR, between the coupling constants at the classical level.

Now we need to calculate the one-loop corrections to the scalar potential. From (3.10) we must calculate

$$\Delta U^{(1)}(\Phi) = -\frac{1}{2} \text{tr}(U^{(1)})^2 + g^2 U^{(1)T} U^{(1)} + 3 \text{tr}(P^2)/2$$

Denoting  $\frac{\partial^2 U}{\partial \phi_j \partial \phi_i}$  by  $U_{ji}$  we have from (3.49)

$$U_{00} = 12\lambda \phi_0^2 + 2\lambda_m \phi_a^2, \quad U_{a0} = 4\lambda_m \phi_0 \phi_a + 3\lambda_d (\phi_a \phi_a)_a$$

$$U_a = 2\lambda_m \phi_0^2 \phi_a + 4\lambda_s \phi_a^2 \phi_a + 3\lambda_d \phi_0 d_{aef} \phi_e \phi_f + 4\lambda_{dd} d_{aef} \phi_e d_{fgh} \phi_g \phi_h$$

$$U_{ba} = (2\lambda_m \phi_o^2 + 4\lambda_s \phi^2) \delta_{ab} + 8\lambda_s \phi_a \phi_b + 6\lambda_d \phi_o d_{abe} \phi_e + 4\lambda_{dd} d_{aef} d_{efg} \phi_f \phi_g \\ + 8\lambda_{dd} d_{aef} \phi_e d_{fgb} \phi_g$$

It is convenient to rewrite  $U_{ba}$ , using (B.7), as

$$U_{ba} = (2\lambda_m \phi_o^2 + 4(\lambda_s + 2\lambda_{dd}/r) \phi^2) \delta_{ab} + 8(\lambda_s - 2\lambda_{dd}/r) \phi_a \phi_b \\ + d_{abe} (6\lambda_d \phi_o \phi_e + 8\lambda_{dd} (\phi_* \phi)_e) + 2\lambda_{dd} \text{tr}(\lambda_a \lambda_e \lambda_b \lambda_f) \phi_o \phi_f$$

Now  $\frac{1}{2} \text{tr}(U'^2) = \frac{1}{2} U_{oo}^2 + (U_{oa})^2 + \frac{1}{2} (U_{ba})^2$ . Using the above we find

$$U_{oo}^2 = 144\lambda_m^2 \phi_o^4 + 4\lambda_m^2 (\phi^2)^2 + 48\lambda_m \lambda_o \phi^2 \phi^2 \\ (U_{oa})^2 = 16\lambda_m^2 \phi_o^2 \phi^2 + 24\lambda_m \lambda_d \phi_o \phi \cdot (\phi_* \phi) + 9\lambda_d^2 (\phi_* \phi)^2$$

and after a lengthy calculation, using the trace formulae (B.3), (B.6) to (B.9) given in Appendix B,

$$(U_{ba})^2 = 4\lambda_m^2 \phi_o^4 + (48\lambda_m \lambda_s - 64\lambda_m \lambda_{dd} + 36(r^2 - 4)\lambda_d^2/r) \phi_o^2 \phi^2 \\ + (144\lambda_s^2 - 128\lambda_s \lambda_{dd}/r + (64 - 384/r^2)\lambda_{dd}^2) (\phi^2)^2 \\ + (96(r^2 - 8)\lambda_d \lambda_{dd}/r + 96\lambda_d \lambda_s) \phi_o \phi \cdot (\phi_* \phi) + ((64r - 960/r)\lambda_{dd}^2 + 192\lambda_{dd} \lambda_s) (\phi_* \phi)^2$$

Next,  $g^2 U'^2_\Phi = g^2 (-r) U_a \phi_a$  since for the singlet  $T^2 = 0$  and for the adjoint  $T^2 = -r$ . Hence

$$g^2 U'^2_\Phi = -2rg^2 \lambda_m \phi_o^2 \phi^2 - 4rg^2 \lambda_s (\phi^2)^2 - 3rg^2 \lambda_d \phi_o \phi \cdot (\phi_* \phi) - 4rg^2 \lambda_{dd} (\phi_* \phi)^2$$

Finally,  $\text{tr}(P^2)$  has no contribution from the singlet since  $T^2 = 0$  and for the adjoint it becomes

$$\text{tr}(P^2) = g^4 (t_{a\phi})_c (t_{b\phi})_c (t_{b\phi})_d (t_{a\phi})_d \\ = g^4 \phi_e \phi_f \phi_g \phi_h f_{ace} f_{bcf} f_{bdg} f_{adh}$$

Using (B.1) and (B.2) we obtain

$$f_{ace}f_{bcf}\phi_e\phi_f = -\text{tr}(\lambda_e\lambda_a\lambda_f\lambda_b)\phi_e\phi_f/4 + \phi^2\delta_{ab}/r + d_{abe}(\phi*\phi)_e/2$$

Substituting this into our expression for  $\text{tr}(P^2)$  and using (B.6) to (B.9) we obtain

$$\text{tr}(P^2) = g^4(2(\phi^2)^2 + r(\phi*\phi)^2/4)$$

Putting all this information together (3.10) gives

$$\begin{aligned} -U^{(1)}(\phi^4) = & ((72\lambda^2 + 2\lambda_m^2)\phi_0^4 \\ & + (2\lambda_m^2 + 72\lambda_s^2 - 64\lambda_s\lambda_{dd}/r + (32 - 192/r^2)\lambda_{dd}^2 + 4r\lambda_sg^2 + 3g^4)(\phi^2)^2 \\ & + (16\lambda_m^2 + 48\lambda\lambda_m + 24\lambda_m\lambda_s - 32\lambda_m\lambda_{dd}/r + 18(r - 4/r)\lambda_d^2 + 2r\lambda_mg^2)\phi_0^2\phi^2 \\ & + (24\lambda_m\lambda_d + 48(r - 8/r)\lambda_d\lambda_{dd} + 48\lambda_d\lambda_s + 3r\lambda_dg^2)\phi_0\phi*\phi \\ & + (9\lambda_d^2 + 32(r - 15/r)\lambda_{dd}^2 + 96\lambda_{dd}\lambda_s + 4r\lambda_{dd}g^2 + 3rg^4/8)(\phi*\phi)^2)/16\pi^2\epsilon \end{aligned} \quad (3.51)$$

If we adopt in addition to (3.7) the renormalised couplings

$$\begin{aligned} \lambda^u &= Z_\lambda Z_{\phi_0}^{-2} \lambda^r \\ \lambda_s^u &= Z_s Z_\phi^{-2} \lambda_s^r \\ \lambda_m^u &= Z_m Z_\phi^{-1} Z_{\phi_0}^{-1} \lambda_m^r \\ \lambda_d^u &= Z_d Z_{\phi_0}^{-1/2} Z_\phi^{-3/2} \lambda_d^r \\ \lambda_{dd}^u &= Z_{dd} Z_\phi^{-2} \lambda_{dd}^r \end{aligned} \quad (3.52)$$

then we have

$$-U^{(1)}(\phi) = Z_\lambda^{(1)}\lambda_{\phi_0}^4 + Z_s^{(1)}\lambda_s(\phi^2)^2 + Z_m^{(1)}\lambda_{m\phi_0}\phi^2 + Z_d^{(1)}\lambda_{d\phi_0}\phi*\phi + Z_{dd}^{(1)}\lambda_{dd}(\phi*\phi)^2$$

and the one-loop renormalisation constants may be read off from (3.51).

The relationships (3.50) amongst unrenormalised parameters imply relationships amongst renormalised parameters. These are consistent with (3.52) provided we have

$$\begin{aligned}
 Z_\lambda^{(1)} - 2Z_{\phi_0}^{(1)} + Z_A^{(1)} &= 0 \\
 Z_s^{(1)} - 2Z_{\underline{\phi}}^{(1)} + Z_A^{(1)} &= 0 \\
 Z_m^{(1)} - Z_\phi^{(1)} - Z_{\phi_0}^{(1)} + Z_A^{(1)} &= 0 \\
 Z_d^{(1)} - 3Z_{\underline{\phi}}^{(1)}/2 - \frac{1}{2}Z_{\phi_0}^{(1)} + Z_A^{(1)} &= 0 \\
 Z_{dd}^{(1)} - 2Z_{\underline{\phi}}^{(1)} + Z_A^{(1)} &= 0
 \end{aligned} \tag{3.53}$$

However after substitution for the renormalised parameters in the renormalisation constants using (3.47), (3.50) and (3.51) we obtain

$$\begin{aligned}
 (72/r^2 l(l+1)+7)rg^2 &\neq 0 \\
 (8(r+8)/rl(l+1)+4r+3rl(l+1))g^2 &\neq 0 \\
 (8(3r^2+7)/r^2 l(l+1)+1)rg^2 &\neq 0 \\
 (24/l(l+1)-2)rg^2 & \\
 (16/l(l+1)+4+3l(l+1)/4)rg^2 &\neq 0.
 \end{aligned} \tag{3.54}$$

The fourth expression appears to be zero for  $l=3$  but it was derived assuming the other coupling constant relationships could be consistently renormalised. Once more we have found the CSDR predictions cannot be renormalised consistently.

In the next section we discuss our results.

### 3.4 DISCUSSION

It has been amply demonstrated in the previous section that the apparent successes of CSDR schemes for model-building are illusory if a consistent renormalisable theory is sought. With the one-loop examples from section 3.3 at hand we move here to a more general discussion of the CSDR constraints and their renormalisation bearing in mind the origin of the CSDR models in higher dimensions.

The coupling and mass relationships of the CSDR scheme generate inconsistencies at the one-loop level because the consequent relations amongst renormalised parameters contain unremovable infinities. If one could work to all orders of the expansion parameter then the relationships might hold. However, at present this cannot be done. Since the relationships are between physical masses and couplings the gauge dependence of our calculations does not matter. One consistent possibility at arbitrary loop order would be an identity between the full renormalised constants. For example

$$Z_\lambda Z_\phi^{-2} = Z_A^{-1}$$

for quartic scalar couplings (see equations (3.21), (3.36), (3.46) and (3.53)). Such identities might be generated by extra symmetries of the CSDR models.

Another view of quantum corrections in CSDR schemes comes from the constraint equations. For a vector gauge field the constraint, as discussed in section 2.2 is

$$L_\xi A_\mu = \partial_\mu W_\xi + g[A_\mu, W_\xi]$$

If we require this constraint to hold under multiplicative renormalisation then we must have  $Z_A^{1/2} = Z_W^{1/2}$  cf (3.7). Now  $W_\xi$  essentially gives the

scalar fields, see (2.4) for example, hence we require

$$Z_A = Z_\phi$$

This is untenable in general.

Another possibility is the existence of more general forms of higher-dimensional action than the  $\int d^d X \sqrt{g} F^{KL} F_{KL}$  usually assumed.

For example in case I [6] the counterpart of this is  $\int d^6 X X^2 F^{KL} F_{KL}$

where  $X^M = (X^\mu, \theta, \bar{\theta})$  runs over four Minkowski and two Grassmann

coordinates, see Section 2.5. In this formalism there is no reason

to exclude a gauge-invariant term of the form  $\int d^6 X X^K F_K^L F_L^M X_M$  with

some arbitrary coefficient. After dimensional reduction the presence of this arbitrary coefficient causes the quartic scalar coupling constant to become arbitrary also. Previously we had  $\lambda = 3g^2/8$  implying a relationship between renormalisation constants if the theory is to be consistently renormalisable. Now there need not be any such relation. Possibly similar generalisations could be made for the other cases considered. In any case the introduction of such gauge invariant terms will remove some of the classical predictions of the CSDR models.

Discussion of CSDR from the higher-dimensional viewpoint raises the question of comparison with the full harmonic expansion mentioned in Section 2.1. This comparison has been made by Palla [13] for a Yang-Mills theory coupled to fermions. The zero-mass spectrum of the full harmonic expansion was found to be greater than that of the CSDR scheme. Perhaps CSDR should be augmented by some consistent dynamical mechanism which includes the effects of some of these fields. Renormalisation is also crucially affected by matter fields which were ignored in the present treatment.

In view of the difficulties raised in this chapter, the utility of CSDR schemes for model-building is questionable. In the next two chapters we make an investigation of the second application of CSDR schemes that is their application to BRST quantisation.



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## 4 BRST QUANTISATION OF ANTISYMMETRIC TENSOR FIELDS

### 4.1 PREAMBLE

In this chapter we apply the formalism of Chapter 2 to the BRST quantisation of rank- $R$  antisymmetric tensor gauge fields [1]. The covariant quantisation of these fields has received many treatments in the past [5-7,9-13]. The rank-2 or Kalb-Ramond field is of interest as a mediator of string interactions [2] and as a member of the minimal set of auxiliary fields for supergravity [3]. The rank-3 field has also been studied in connection with supergravity [4] and the  $U(1)$  problem [5].

For the readers convenience we commence in Section 4.2 with the BRST quantisation of the Kalb-Ramond field. Following the method described in Chapter 2 we derive the constraint equation for the fields and then for the gauge parameter. Expanding the gauge parameter as a function of the Grassmann coordinates we solve for the components. These solutions are then used in the field constraint to find the components of the superfield. We leave any discussion of our results until after the rank- $R$  case is treated.

In Section 4.3 we perform the same steps for the general rank- $R$  antisymmetric tensor gauge field. Naturally these reduce for  $R = 2$  to the results of the previous section. The reader who wishes to skip the details of the calculation is guided directly to the main results. We then compare our ghost-spectrum and degree-of-freedom count with those obtained by other authors.

Finally, in Section 4.4 we construct a six-dimensional, BRST invariant action for the Kalb-Ramond field. We then substitute our solutions to the field constraint into the action and reduce it to its four-dimensional form. We compare this action also with the work

of previous authors.

#### 4.2 CONSTRAINTS AND SOLUTIONS FOR THE KALB-RAMOND FIELD

We take our fields  $V_{MN}(x, \theta, \bar{\theta}) = -[MN]V_{NM}(x, \theta, \bar{\theta})$  to be in an irreducible representation of  $OSp(4/2)$ . Our space-time symmetry is  $M^4XS/R$  where  $S = Sp(2) \wedge T_2$  and  $R = Sp(2)$ . Then the requirement that our fields be  $S$ -invariant leads us to a constraint (4.2) on the fields. We derive this constraint as follows.

Suppose the infinitesimal action of the group  $S$ , with generators  $J_{\bar{a}}^a$  satisfying  $[J_{\bar{a}}^a, J_{\bar{b}}^b] = C_{\bar{a}\bar{b}}^{\bar{c}} J_{\bar{c}}^{\bar{c}}$ , on the coordinates is

$$x^M \rightarrow x^M + \rho^M$$

$$\text{with } \rho^M = \rho^{\bar{a}} \xi_{\bar{a}}^M.$$

$S$  also induces a variation of the antisymmetric field. To find this action we write

$$V_{MN} = T_M^U U_N - [MN] T_N^U U_M$$

then the vectors transform as

$$T'_M(x') = \frac{\partial x^K}{\partial x'^M} T_K(x)$$

hence  $V_{MN}$  transforms as

$$\begin{aligned} V'_{MN}(x') &= \frac{\partial x^K}{\partial x'^M} T_K(x) \frac{\partial x^L}{\partial x'^N} U_L(x) - [MN] \frac{\partial x^L}{\partial x'^N} T_L(x) \frac{\partial x^K}{\partial x'^M} U_K(x) \\ &= -[KN] \frac{\partial x^K}{\partial x'^M} \frac{\partial x^L}{\partial x'^N} (T_L U_K - [KL] T_K U_L) \\ &= -[KN] (\delta_M^K - \partial_M \rho^K) (\delta_N^L - \partial_N \rho^L) V_{LK}(x) \\ &= V_{MN}(x) + [KN] \partial_M \rho^K V_{NK} + [MN] \partial_N \rho^L V_{LM} \end{aligned}$$

A Taylor expansion of  $V'_{MN}$  around  $x$  gives also

$$V'_{MN}(x') = V_{MN}(x) + \rho^L \partial_L V_{MN}(x)$$

Then the Lie derivative of  $V_{MN}$  is given by

$$\begin{aligned} L_\rho V_{MN} &= V'_{MN}(X) - V_{MN}(X) \\ &= -\partial_M \rho^K V_{KN} + [MN] \partial_N \rho^L V_{LM} - \rho^L \partial_L V_{MN} \end{aligned} \quad (4.1)$$

The field  $V_{MN}$  is S-invariant if the action of S is compensated by a gauge transformation with gauge parameter

$$\begin{aligned} W_N^\rho &= \rho^{\bar{a}} \bar{W}_{\bar{a}N} \\ \delta_\rho V_{MN} &= \partial_M W_N^\rho - [MN] \partial_N W_M^\rho \end{aligned}$$

The constraint on the field  $V_{MN}$  is then given by

$$-\partial_M \rho^K V_{KN} + [MN] \partial_N \rho^L V_{LM} - \rho^L \partial_L V_{MN} = \partial_M W_N^\rho - [MN] \partial_N W_M^\rho \quad (4.2)$$

Now the commutator of two Lie derivatives is another Lie derivative.

Applying this to the Lie derivative of  $V_{MN}$  we obtain a constraint (4.4) on the gauge parameter. We require

$$\begin{aligned} [L_\sigma, L_\rho] V_{MN} &= L_{[\sigma, \rho]} V_{MN} \\ &= \partial_M W_N^{[\sigma, \rho]} - [MN] \partial_N W_M^{[\sigma, \rho]} \end{aligned} \quad (4.3)$$

where  $[\sigma, \rho]^{\bar{a}} = -\sigma^{\bar{c}} \rho^{\bar{b}} \bar{C}_{\bar{b}\bar{c}}^{\bar{a}}$ .

We have from (4.1) and (4.2)

$$\begin{aligned} L_\rho L_\sigma V_{MN} &= L_\rho (\partial_M W_N^\sigma - [MN] \partial_N W_M^\sigma) \\ &= -\partial_M \rho^K (\partial_K W_N^\sigma - [KN] \partial_N W_K^\sigma) + [MN] \partial_N \rho^K (\partial_K W_M^\sigma - [KM] \partial_M W_K^\sigma) \\ &\quad - \rho^K \partial_K (\partial_M W_N^\sigma - [MN] \partial_N W_M^\sigma) \end{aligned}$$

Hence we find

$$[L_\sigma, L_\rho] V_{MN} = \partial_M L_\sigma W_N^\rho - \partial_M L_\rho W_N^\sigma - [MN] \partial_N L_\sigma W_M^\rho + [MN] \partial_N L_\rho W_M^\sigma$$

where we have used

$$L_\sigma W_N^\rho = -\sigma^L \partial_L W_N^\rho - (\partial_N \sigma^L) W_L^\rho$$

since  $W_N^\rho$  is a vector.

Then the requirement (4.3) implies

$$\partial_M (L_O W_N^\rho - L_\rho W_N^\sigma) - [MN] \partial_N (L_O W_M^\rho - L_\rho W_M^\sigma) = \partial_M W_N^{[\sigma, \rho]} - [MN] \partial_N W_M^{[\sigma, \rho]}$$

We then take the constraint on the gauge parameter to be

$$L_O W_N^\rho - L_\rho W_N^\sigma = W_N^{[\sigma, \rho]} \quad (4.4)$$

although a more general solution of the form  $L_O W_N^\rho - L_\rho W_N^\sigma - W_N^{[\sigma, \rho]} = \partial_N Z$  is possible.

We can now solve these constraints for the case of interest to BRST quantisation namely  $M^4_{XS/R}$  where  $S = Sp(2, R) \wedge T_2$  and  $R = Sp(2, R)$ . The Killing vectors  $\xi_a^M$  and the structure constants for this group have been given in (2.23) and (2.22) respectively. We shall solve the gauge parameter constraint first then use these solutions to solve the field constraint.

To solve (4.4) for the gauge parameter we need to consider three cases (i)  $(\rho^{\bar{a}}, \sigma^{\bar{b}}) = (\epsilon^p, \rho^q)$ , (ii)  $(\epsilon^p, \tau^{mn}/2)$  and finally (iii)  $(\tau^{mn}/2, \tau^{kl}/2)$ .

i) For this case we have  $c_{pq}^M = 0$  from (2.22) hence (4.4) becomes

$$0 = -\epsilon^p \xi_p^L \partial_L (\rho^q W_{qN}) - \partial_N (\epsilon^p \xi_p^L) \rho^q W_{qL} + \rho^q \xi_q^L \partial_L (\epsilon^p W_{pN}) + \partial_N (\rho^q \xi_q^L) \epsilon^p W_{pL}$$

Substituting for the Killing vectors this reduces to

$$0 = \partial_p W_{qN} + \partial_q W_{pN} \quad (4.5)$$

The solution to this may be written in the form

$$W_{qN} = \partial_q F_N \quad (4.6)$$

where  $F_N(x, \theta=0) = 0$

ii) The gauge parameter constraint becomes

$$\begin{aligned} & -\epsilon^p \xi_p^L \partial_L (\tfrac{1}{2} \tau^{mn} W_{mnN}) + \tfrac{1}{2} \tau^{mn} \xi_{mn}^L \partial_L (\epsilon^p W_{pN}) + \tfrac{1}{2} \partial_N (\tau^{mn} \xi_{mn}^L) \epsilon^p W_{pL} \\ & = -\tfrac{1}{2} \epsilon^p \tau^{mn} \eta_{np} W_{mN} - \tfrac{1}{2} \epsilon^p \tau^{mn} \eta_{mp} W_{nN} \end{aligned}$$

Using the Killing vectors we then solve for (a)  $N = \mu$  and (b)  $N = 1$ .

a) The constraint gives

$$-\partial_p W_{mn\mu} + \theta_m \partial_n W_{p\mu} + \theta_n \partial_m W_{p\mu} = -\eta_{np} W_{m\mu} - \eta_{mp} W_{n\mu} \quad (4.7)$$

It is convenient here to introduce the operators  $\Sigma_{mn}$  and  $\Sigma_{mn}^0$ .  $\Sigma_{mn}$  acts only on  $Sp(2)$  indices. Its action is given by

$$\begin{aligned} \Sigma_{mn} T_{p_1 \dots p_q} &= \eta_{mp_1} T_{np_2 \dots p_q} + \eta_{np_1} T_{mp_2 \dots p_q} + \dots + \eta_{mp_q} T_{p_1 \dots p_{q-1} n} \\ &\quad + \eta_{np_q} T_{p_1 \dots p_{q-1} m} \end{aligned} \quad (4.8)$$

where the  $p_i$  are  $Sp(2)$  indices. Note also if  $T_{p_1 \dots p_i \dots p_j \dots p_q} = T_{p_1 \dots p_j \dots p_i \dots p_q}$  then the action of  $\Sigma_{mn}$  is simply

$$\Sigma_{mn} T_{p_1 \dots p_q} = \sum_{i=1}^q \eta_{mp_i} T_{np_1 \dots \hat{p}_i \dots p_q} + \eta_{np_i} T_{mp_1 \dots \hat{p}_i \dots p_q} \quad (4.9)$$

where  $\hat{p}_i$  denotes the absence of the index  $p_i$ .  $\Sigma_{mn}^0$  is given by

$$\Sigma_{mn}^0 = \theta_m \partial_n + \theta_n \partial_m \quad (4.10)$$

We then construct an operator  $\Sigma'_{mn}$

$$\Sigma'_{mn} = \Sigma_{mn} + \Sigma_{mn}^0 \quad (4.11)$$

with properties

$$[\Sigma'_{mn}, \Sigma'_{kl}] = \eta_{mk} \Sigma'_{nl} + \eta_{ml} \Sigma'_{kn} + \eta_{nk} \Sigma'_{ml} + \eta_{nl} \Sigma'_{km} \quad (4.12)$$

$$[\Sigma'_{mn}, \partial_p] = 0 \quad (4.13)$$

We can rewrite (4.7) using these operators as

$$-\partial_p W_{mn\mu} = -\Sigma'_{mn} W_{p\mu}$$

Using (4.6), the solution for  $W_{p\mu}$ , this becomes

$$\partial_p W_{mn\mu} = \partial_p \Sigma'_{mn} F_\mu$$

since  $\Sigma'_{mn}$  and  $\partial_p$  commute. The solution to this may be written

$$W_{mn\mu} = A_{mn\mu} + \Sigma'_{mn} F_\mu \quad (4.14)$$

where  $A_{mn\mu}$  has no  $\theta$  dependence.

b) If  $N = 1$  we obtain

$$-\partial_p W_{mnl} + \theta_m \partial_n W_{pl} + \theta_n \partial_m W_{pl} + \eta_{ml} W_{pn} + \eta_{nl} W_{pm} = -\eta_{np} W_{ml} - \eta_{mp} W_{nl}$$

which we can rewrite as

$$-\partial_p W_{mnl} = -\Sigma'_{mn} W_{pl}$$

Using the solution (4.6) for  $W_{pl}$  this becomes

$$\partial_p W_{mnl} = \partial_p \Sigma'_{mn} F_l$$

with solution

$$W_{mnl} = A_{mnl} + \Sigma'_{mn} F_l \quad (4.15)$$

where  $A_{mnl}$  has no  $\theta$  dependence.

iii) In this case the constraint becomes

$$\begin{aligned} & -\epsilon_{mn}^M \partial_M W_{klN} - \partial_N (\epsilon_{mn}^M) W_{klM} + \epsilon_{kl}^M \partial_M W_{mnN} + \partial_N (\epsilon_{kl}^M) W_{mnM} \\ & = \eta_{nk} W_{mlN} + \eta_{ml} W_{knN} + \eta_{mk} W_{nlN} + \eta_{nl} W_{kmN} \end{aligned} \quad (4.16)$$

Once more we solve for the cases (a)  $N = \mu$  and (b)  $N = q$ .

a) We have

$$-(\theta_m \partial_n + \theta_n \partial_m) W_{kl\mu} + (\theta_k \partial_l + \theta_l \partial_k) W_{mn\mu} = \eta_{nk} W_{ml\mu} + \eta_{ml} W_{kn\mu} + \eta_{mk} W_{nl\mu} + \eta_{nl} W_{km\mu}$$

which we can rewrite as

$$\Sigma'_{kl} W_{mn\mu} - \Sigma'_{mn} W_{kl\mu} = \eta_{km} W_{ln\mu} + \eta_{kn} W_{ml\mu} + \eta_{lm} W_{kn\mu} + \eta_{ln} W_{mk\mu} \quad (4.17)$$

Using the solution (4.12) for  $W_{mn\mu}$  we can split (4.15) into two parts

$$\Sigma_{kl} A_{mn\mu} - \Sigma_{mn} A_{kl\mu} = \eta_{km} A_{ln\mu} + \eta_{kn} A_{ml\mu} + \eta_{lm} A_{kn\mu} + \eta_{ln} A_{mk\mu} \quad (4.18)$$

and

$$[\Sigma'_{kl}, \Sigma'_{mn}] F_\mu = (\eta_{km} \Sigma'_{ln} + \eta_{kn} \Sigma'_{ml} + \eta_{lm} \Sigma'_{kn} + \eta_{ln} \Sigma'_{mk}) F_\mu$$

which is simply an identity by (4.12).

Now (4.18) is simply  $\Sigma_{kl} A_{mn\mu} = 0$ . Introducing the operator

$C_2 = \Sigma^{kl} \Sigma_{kl}$ , whose action is calculated in Appendix C,

we can write this as  $C_2 A_{mn\mu} = 0$ .

Then equation (C.2) gives us  $C_2 A_{mn\mu} = -16 A_{mn\mu}$  hence the solution to (4.18) is

$$A_{mn\mu} = 0. \quad (4.19)$$

b) For  $N = q$  (4.16) becomes

$$\Sigma'_{kl} W_{mnq} - \Sigma'_{mn} W_{klq} = \eta_{km} W_{lnq} + \eta_{kn} W_{mlq} + \eta_{lm} W_{knq} + \eta_{ln} W_{mkq}$$

Using the solution (4.15) for  $W_{mnq}$  this becomes

$$\Sigma_{kl} A_{mnq} - \Sigma_{mn} A_{klq} = \eta_{km} A_{lnq} + \eta_{lm} A_{knq} + \eta_{ln} A_{mkq} + \eta_{kn} A_{mlq} \quad (4.20)$$

and an identity once more for  $F_q$ .

To solve (4.20) we operate on both sides with  $\Sigma^{mn}$  then it becomes

$$C_2 A_{klq} = \Sigma^{mn} (\eta_{kq} A_{mnl} + \eta_{lq} A_{mnk}) \quad (4.21)$$

Now we decompose  $A_{klq}$  into

$$A_{klq} = \alpha_1 A^{<3>}_{klq} + \alpha_2 A^{<1>}_{klq}$$

where

$$A^{<3>}_{klq} = A_{klq} + A_{lqk} + A_{qkl}$$

$$A^{<1>}_{klq} = \Sigma_{kl} A^r_{rq}$$

$$= -2A_{klq} + A_{lqk} + A_{qkl} \quad (4.22)$$

Hence  $\alpha_1 = 1/3 = -\alpha_2$ .

Now Appendix C equation (C.2) gives

$$C_2 A_{klq} = -14A_{klq} - 8A_{lqk} - 8A_{qkl}$$

Using (4.22) we can rewrite this as

$$C_2 A_{klq} = 2A^{<1>}_{klq} - 10A^{<3>}_{klq} \quad (4.23)$$

We can express the right-hand side of (4.21) in terms of

$A^{<1>}_{klq}$  and  $A^{<3>}_{klq}$  by



$$\begin{aligned}
\Sigma^{mn}(\eta_{kq}A_{mnl} + \eta_{lq}A_{mnk}) &= 2(\eta_{kq}\eta^{nr}A_{lnr} + \eta_{lq}\eta^{nr}A_{knr}) \\
&= -4A_{klq} + 2A_{lqk} + 2A_{kql} \\
&= 2A^{<1>}_{klq}
\end{aligned} \tag{4.24}$$

From (4.23) and (4.24) we find only  $A^{<1>}_{klq} \neq 0$ . Letting  $A_{rq}^r = 3B_q$  we have from (4.22)

$$A_{klq} = -\Sigma_{kl}B_q \tag{4.25}$$

Finally performing a  $\theta$  expansion of  $F_N$

$$F_N = \theta^S A_{sN} + \frac{1}{2}\theta^2 \lambda_N$$

we can write our solutions (4.6), (4.14) and (4.15) as

$$W_{qN} = A_{qN} + \theta \lambda_N \tag{4.26}$$

$$W_{mn\mu} = \theta^S \Sigma_{mn} A_{s\mu} \tag{4.27}$$

$$W_{mnl} = A_{mnl} + \theta^P \Sigma_{mn} A_{pl} + \frac{1}{2}\theta^2 \Sigma_{mn} \lambda_l \tag{4.28}$$

where  $A_{mnl} = -\Sigma_{mn}B_l$  from (4.25).

Having solved the gauge parameter constraint we can now use the solutions to solve the field constraint.

The field constraint (4.2) can be considered in six parts:

- (i)  $(\rho^{\bar{a}}, M, N) = (\epsilon^n, \mu, \nu)$ , (ii)  $(\epsilon^n, \mu, q)$ , (iii)  $(\epsilon^n, l, q)$ , (iv)  $(\frac{1}{2}\tau^{mn}, \mu, \nu)$ ,  
(v)  $(\frac{1}{2}\tau^{mn}, \mu, l)$  and (vi)  $(\frac{1}{2}\tau^{mn}, l, q)$ .

i) The constraint is

$$-\epsilon^n \xi_n^L \partial_L V_{\mu\nu} = \partial_\mu (\epsilon^n W_{n\nu}) - \partial_\nu (\epsilon^n W_{n\mu})$$

Using the solution for  $W_{qN}$  (4.26) this gives

$$\partial_n V_{\mu\nu} = \partial_\nu A_{n\mu} - \partial_\mu A_{nv} - \theta_n (\partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu)$$

The most general solution to this is

$$V_{\mu\nu} = v_{\mu\nu} + \theta^P (\partial_\nu A_{p\mu} - \partial_\mu A_{pv}) + \frac{1}{2}\theta^2 (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) \tag{4.29}$$

ii) For this case the constraint yields

$$\partial_n V_{\mu q} = -\partial_\mu W_{nq} - \partial_q W_{n\mu}$$

By (4.26) this becomes

$$\partial_n V_{\mu q} = -\partial_\mu A_{nq} - \theta \partial_n \lambda_q - \eta_{nq} \lambda_\mu$$

whose solution is

$$V_{\mu q} = c_{\mu q} - \theta^m (\eta_{mq} \lambda_\mu + \partial_\mu A_{mq}) - \frac{1}{2} \theta^2 \partial_\mu \lambda_q \quad (4.30)$$

iii) In this case we obtain

$$\partial_n V_{lq} = \partial_l W_{nq} + \partial_q W_{nl}$$

Then (4.26) gives us

$$\partial_n V_{lq} = \eta_{nl} \lambda_q + \eta_{nq} \lambda_l$$

and the solution to this is

$$V_{lq} = c_{lq} + \theta^m (\eta_{ml} \lambda_q + \eta_{mq} \lambda_l) \quad (4.31)$$

iv) The field constraint becomes

$$-\theta \partial_m \partial_n V_{\mu\nu} - \theta \partial_n \partial_m V_{\mu\nu} = \partial_\mu W_{mn\nu} - \partial_\nu W_{mn\mu}$$

Substituting the solution for  $W_{mn\mu}$  (4.27) and for  $V_{\mu\nu}$  (4.29) we find this to be trivially satisfied.

v) For this case we obtain

$$\eta_{ml} V_{n\mu} + \eta_{nl} V_{m\mu} - \theta \partial_m \partial_n V_{\mu l} - \theta \partial_n \partial_m V_{\mu l} = \partial_\mu W_{mnl} - \partial_l W_{mn\mu}$$

Using the solution (4.28) since  $V_{n\mu} = -V_{\mu n}$  and (4.30) this becomes to zeroth order in  $\theta$

$$-\sum_{mn} A_{l\mu} = \eta_{lm} c_{\mu n} + \eta_{ln} c_{\mu m} - \eta_{lm} \partial_\mu B_n - \eta_{ln} \partial_\mu B_m$$

which we write as

$$\sum_{mn} c_{l\mu} = -\sum_{mn} A_{l\mu} - \sum_{mn} \partial_\mu B_l$$

hence

$$c_{l\mu} = -A_{l\mu} - \partial_\mu B_l \quad (4.32)$$

The first and second orders in  $\theta$  are trivially satisfied.

vi) The constraint becomes

$$-\eta_{ml}V_{nq} - \eta_{nl}V_{mq} - \eta_{mq}V_{nl} - \eta_{nq}V_{ml} - \theta_m \partial_n V_{lq} - \theta_n \partial_m V_{lq} = \partial_l W_{mnq} + \partial_q W_{mnl}$$

The solutions (4.28) and (4.31) then imply

$$\begin{aligned} & \eta_{lm}c_{nq} + \eta_{ln}c_{mq} + \eta_{qm}c_{nl} + \eta_{qn}c_{ml} + \theta_q(-\eta_{lm}\lambda_n - \eta_{ln}\lambda_m) - \theta_l\eta_{qm}\lambda_n - \theta_l\eta_{qn}\lambda_m \\ &= \Sigma_{mn}A_{lq} + \Sigma_{mn}A_{ql} + \theta_l\Sigma_{mn}\lambda_q + \theta_q\Sigma_{mn}\lambda_l \end{aligned}$$

At zeroth order in  $\theta$  this gives

$$-\Sigma_{mn}c_{lq} = \Sigma_{mn}A_{lq} + \Sigma_{mn}A_{ql}$$

hence

$$c_{lq} = -A_{lq} - A_{ql} \quad (4.33)$$

First and second order equations in  $\theta$  are again identities.

Before writing the solutions in their final form it is convenient to define

$$A'_{l\mu} = A_{l\mu} + \partial_\mu B_l$$

Then the solutions from (4.29) to (4.33) are

$$V_{\mu\nu} = v_{\mu\nu} + \theta^m(\partial_\nu A'_{m\mu} - \partial_\mu A'_{m\nu}) + \frac{1}{2}\theta^2(\partial_\nu\lambda_\mu - \partial_\mu\lambda_\nu)$$

$$V_{\mu l} = A'_{l\mu} + \theta^m(\eta_{lm}\lambda_\mu - \partial_\mu A_{ml}) - \frac{1}{2}\theta^2\partial_\mu\lambda_l$$

$$V_{lq} = -A_{lq} - A_{ql} + \theta^m(\eta_{ml}\lambda_q + \eta_{mq}\lambda_l)$$

which we can clearly write, dropping the prime on  $A'_{p\mu}$ , as

$$V_{MN} = X_{MN} - \partial_M(\theta^q A_{qN} + \frac{1}{2}\theta^2\lambda_N) - [MN]\partial_N(\theta^q A_{qM} + \frac{1}{2}\theta^2\lambda_M) \quad (4.34)$$

where  $X_{\mu\nu} = v_{\mu\nu}$  and  $X_{\mu l} = X_{lq} = 0$ . We will return to this form later.

We now write the solutions to the field constraint in their final form. To do so we decompose  $A_{ql}$  into  $A_{ql} = -\frac{1}{2}c_{lq} + \frac{1}{2}\eta_{lq}r$  and define

$$\lambda'_\mu = \lambda_\mu - \frac{1}{2}\partial_\mu A^q_q$$

then our solutions, dropping the prime on  $\lambda'_\mu$ , are

$$\begin{aligned}
 V_{\mu\nu} &= v_{\mu\nu} + \theta^m (\partial_\nu c_{\mu m} - \partial_\mu c_{\nu m}) + \frac{1}{2} \theta^2 (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) \\
 V_{\mu 1} &= c_{\mu 1} + \theta^m (\eta_{1m} \lambda_\mu + \frac{1}{2} \partial_\mu c_{1m}) - \frac{1}{2} \theta^2 \partial_\mu \lambda_1 \\
 V_{1q} &= c_{1q} + \theta^m (\eta_{mq} \lambda_1 + \eta_{mq} \lambda_q)
 \end{aligned} \tag{4.35}$$

Before we discuss these results we perform the corresponding calculation for the rank-R anti-symmetric tensor gauge fields.

#### 4.3 CONSTRAINTS AND SOLUTIONS FOR HIGHER RANK ANTI-SYMMETRIC FIELDS

We write the superfield as  $V_{M_1 \dots M_R}(x, \theta, \bar{\theta})$  where for convenience we take the first  $q$  indices to be odd and the last  $R-q$  even.

The superfield in this form has the following symmetry properties

$$V_{M_1 \dots M_i \dots M_j \dots M_R} = \begin{cases} V_{M_1 \dots M_j \dots M_i \dots M_R}, & 1 \leq i < j \leq q \\ -V_{M_1 \dots M_j \dots M_i \dots M_R}, & q+1 \leq i < j \leq R \end{cases} \tag{4.36}$$

We now derive the constraint equation resulting from the requirement that the superfields be S-invariant. The field constraint is given in equation (4.40) and the gauge parameter constraint in equation (4.44) should the reader wish to skip the details of the calculation.

A rank  $R$  tensor may be written as a linear combination of terms which are a product of  $R$  rank one tensors. Thus to find the transformation of a rank- $R$  tensor under a change of coordinates one need only consider the transformation of such a product.

A product of  $R$  rank one tensors transforms, under a change of coordinates, as

$$\begin{aligned}
 V'_{M_1} \dots V'_{M_R}(X') &= \frac{\partial X^{N_1}}{\partial X'^{M_1}} \dots \frac{\partial X^{N_R}}{\partial X'^{M_R}} V_{N_1} \dots V_{N_R} \\
 &= \frac{\partial X^{N_1}}{\partial X'^{M_1}} \dots \frac{\partial X^{N_R}}{\partial X'^{M_R}} (\prod_{1 \leq i < j \leq R} [N_i N_j][N_i M_j]) V_{N_1} \dots V_{N_R}
 \end{aligned}$$

Hence  $V_{M_1 \dots M_R}$  transforms as

$$V'_{M'_1 \dots M'_R}(X') = \frac{\partial X^{N_1}}{\partial X'^{M'_1}} \dots \frac{\partial X^{N_R}}{\partial X'^{M'_R}} (\prod_{1 \leq i < j \leq R} [N_i N_j][N_i M'_j]) V_{N_1} \dots V_{N_R}$$

Suppose the infinitesimal action of a group  $S$ , with generators  $J_a^-$  satisfying  $[J_a^-, J_b^-] = C_{ab}^c J_c^-$ , on the coordinates is  $X'^M = X^M + \rho^M$ ,

where  $\rho^M = \rho^a \bar{\epsilon}_a^M$  and the  $\bar{\epsilon}_a^M$  are the Killing vectors. Then this

induces a transformation on  $V_{M_1 \dots M_R}$

$$\begin{aligned}
 &(\delta_{M_1}^{N_1} - \partial_{M_1} \rho^{N_1}) \dots (\delta_{M_R}^{N_R} - \partial_{M_R} \rho^{N_R}) (\prod_{1 \leq i < j \leq R} [N_i N_j][N_i M_j]) V_{N_1} \dots V_{N_R} \\
 &= \sum_{k=1}^R -\partial_{M_k} \rho^{N_k} (\prod_{i < k} [M_i N_k][M_i M_k]) V_{M_1 \dots M_{k-1} N_k M_{k+1} \dots M_R} \quad (4.37)
 \end{aligned}$$

Now our rank  $R$  fields satisfy  $V_{M_1 \dots M_R} = \prod_{i=1}^R (-[M_i M_k]) V_{M_k M_1 \dots \hat{M}_k \dots M_R}$  where  $\hat{M}_k$  indicates the subscript  $M_k$  should be omitted, hence equation (4.37) becomes

$$\begin{aligned}
 &\sum_{k=1}^R -\partial_{M_k} \rho^{N_k} (-1)^{k-1} (\prod_{i < k} [M_i M_k]) V_{N_k M_1 \dots \hat{M}_k \dots M_R} \\
 &= \sum_{k=1}^q \partial_{M_k} \rho^{N_k} V_{N_k M_1 \dots \hat{M}_k \dots M_R} - \sum_{k=q+1}^R \partial_{M_k} \rho^{N_k} (-1)^{k-1} V_{N_k M_1 \dots \hat{M}_k \dots M_R} \\
 &= -\sum_{k=1}^R S(k) (\partial_{M_k} \rho^{N_k}) V_{N_k M_1 \dots \hat{M}_k \dots M_R}
 \end{aligned}$$

where

$$\begin{aligned}
 S(k) &= 1 & 1 \leq k \leq q \\
 &(-1)^{k-1} & q+1 \leq k \leq R
 \end{aligned}$$

A Taylor expansion of  $V_{M_1 \dots M_R}$  to first order in  $\rho^N$  gives

$$V'_{M_1 \dots M_R}(X') = V'_{M_1 \dots M_R}(X) + \rho^N \partial_N V_{M_1 \dots M_R}(X)$$

Then the Lie derivative of  $V_{M_1 \dots M_R}$  is

$$L_\rho V_{M_1 \dots M_R} = - \sum_{k=1}^R S(k) (\partial_{M_k} \rho^N) V_{NM_1 \dots \hat{M}_k \dots M_R} - \rho^N \partial_N V_{M_1 \dots M_R} \quad (4.38)$$

$V_{M_1 \dots M_R}$  is said to be  $S$ -invariant if the action of  $S$  can be compensated by

$$\delta_\rho V_{M_1 \dots M_R} = \sum_{k=1}^R S(k) \partial_{M_k} W^\rho_{M_1 \dots \hat{M}_k \dots M_R} \quad (4.39)$$

This requirement gives a constraint on the fields  $V_{M_1 \dots M_R}$  that is

$$- \sum_{k=1}^R S(k) (\partial_{M_k} \rho^N) V_{NM_1 \dots \hat{M}_k \dots M_R} - \rho^N \partial_N V_{M_1 \dots M_R} = \sum_{k=1}^R S(k) \partial_{M_k} W^\rho_{M_1 \dots \hat{M}_k \dots M_R} \quad (4.40)$$

Again if we require that the action induced on  $V_{M_1 \dots M_R}$  by  $S$  acting twice on the coordinates should also be compensated by a transformation of the form (4.39) then

$$\begin{aligned} [L_\sigma, L_\rho] V_{M_1 \dots M_R} &= L_{[\sigma, \rho]} V_{M_1 \dots M_R} \\ &= \sum_{k=1}^R S(k) \partial_{M_k} W^{[\sigma, \rho]}_{M_1 \dots \hat{M}_k \dots M_R} \end{aligned} \quad (4.41)$$

where  $[\sigma, \rho]^\bar{c} = -\sigma^\bar{a} \rho^\bar{b} C_{\bar{b}\bar{a}}^\bar{c}$  and we obtain a constraint on  $W_{M_1 \dots \hat{M}_k \dots M_R}$ ,

$$\begin{aligned} &\sum_{k=1}^R S(k) \partial_{M_k} (L_\sigma W^\rho_{M_1 \dots \hat{M}_k \dots M_R} - L_\rho W^\sigma_{M_1 \dots \hat{M}_k \dots M_R}) \\ &= \sum_{k=1}^R S(k) \partial_{M_k} W^{[\sigma, \rho]}_{M_1 \dots \hat{M}_k \dots M_R} \end{aligned} \quad (4.42)$$

To derive (4.42) we start with

$$L_\sigma L_\rho V_{M_1 \dots M_R} = L_\sigma \left( \sum_{j=1}^R S(j) \partial_{M_j} W^\rho_{M_1 \dots \hat{M}_j \dots M_R} \right)$$

$$\begin{aligned}
&= \sum_{1 \leq i < j \leq R} S(i) (\partial_{M_i} \sigma^N) [M_i M_j] [NM_j] S(j) \partial_{M_j} W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho} \\
&- \sum_{1 \leq j < i \leq R} S(i) (\partial_{M_i} \sigma^N) [NM_j] S(j) \partial_{M_j} W_{NM_1 \dots \hat{M}_j \dots \hat{M}_i \dots M_R}^{\rho} \\
&- \sum_{i=1}^R S(i) (\partial_{M_i} \sigma^N) \partial_N W_{M_1 \dots \hat{M}_i \dots M_R}^{\rho} - \sigma^N \partial_N \sum_{j=1}^R S(j) \partial_{M_j} W_{M_1 \dots \hat{M}_j \dots M_R}^{\rho}
\end{aligned} \tag{4.43}$$

Now we have, irrespective of whether  $i > j$  or  $i < j$ ,

$$\begin{aligned}
&-(\partial_{M_i} \sigma^N) \partial_{M_j} W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho} [NM_j] = -\partial_{M_i} ((\partial_{M_j} \sigma^N) W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}) \\
&+ \partial_{M_i} \partial_{M_j} (\sigma^N W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}) - [NM_i] [NM_j] \sigma^N \partial_{M_i} \partial_{M_j} W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}
\end{aligned}$$

Using this in (4.43) we find

$$\begin{aligned}
L_{\sigma}^L L_{\rho} V_{M_1 \dots M_R} &= - \sum_{1 \leq i < j \leq R} S(i) [M_i M_j] S(j) [-\partial_{M_i} ((\partial_{M_j} \sigma^N) W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}) \\
&+ \partial_{M_i} \partial_{M_j} (\sigma^N W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}) - [NM_i] [NM_j] \sigma^N \partial_{M_i} \partial_{M_j} W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}] \\
&+ \sum_{1 \leq j < i \leq R} S(i) S(j) [-\partial_{M_i} ((\partial_{M_j} \sigma^N) W_{NM_1 \dots \hat{M}_j \dots \hat{M}_i \dots M_R}^{\rho}) \\
&+ \partial_{M_i} \partial_{M_j} (\sigma^N W_{NM_1 \dots \hat{M}_j \dots \hat{M}_i \dots M_R}^{\rho}) - [NM_i] [NM_j] \sigma^N \partial_{M_i} \partial_{M_j} W_{NM_1 \dots \hat{M}_j \dots \hat{M}_i \dots M_R}^{\rho}] \\
&- \sum_{i=1}^R S(i) \partial_{M_i} (\sigma^N \partial_N W_{M_1 \dots \hat{M}_i \dots M_R}^{\rho})
\end{aligned}$$

Now  $\partial_{M_i} \partial_{M_j} = [M_i M_j] \partial_{M_j} \partial_{M_i}$  so this becomes

$$\begin{aligned}
L_{\sigma}^L L_{\rho} V_{M_1 \dots M_R} &= \sum_{1 \leq i < j \leq R} S(i) \partial_{M_i} (S(j) [M_i M_j] (\partial_{M_j} \sigma^N) W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}) \\
&- \sum_{1 \leq j < i \leq R} S(i) \partial_{M_i} (S(j) (\partial_{M_j} \sigma^N) W_{NM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}^{\rho}) \\
&- \sum_{i=1}^R S(i) \partial_{M_i} (\sigma^N \partial_N W_{M_1 \dots \hat{M}_i \dots M_R}^{\rho}) \\
&= \sum_{i=1}^R S(i) \partial_{M_i} (L_{\sigma} W_{M_1 \dots \hat{M}_i \dots M_R}^{\rho})
\end{aligned}$$

Hence

$$[L_\sigma, L_\rho] V_{M_1 \dots M_R} = \sum_{i=1}^R S(i) \partial_{M_i} (L_\sigma W^\rho_{M_1 \dots \hat{M}_i \dots M_R} - L_\rho W^\sigma_{M_1 \dots \hat{M}_i \dots M_R})$$

substituting this in (4.41) gives us the constraint (4.42) on

$$W_{M_1 \dots \hat{M}_k \dots M_R}$$

We then take this constraint to have the form

$$L_\sigma W^\rho_{M_1 \dots M_R} - L_\rho W^\sigma_{M_1 \dots M_R} = W^{[\sigma, \rho]}_{M_1 \dots M_R} \quad (4.44)$$

Again a more general solution is possible.

We now solve the constraints for the case of interest to BRST quantisation. We take  $M^4 \times S/R$  to be  $M^4 \times (Sp(2, R) \wedge T_2) / Sp(2, R)$ . The appropriate Killing vectors and structure constants have been given already in equations (2.23) and (2.22) respectively. The solutions to the gauge parameter constraint will be used to solve the field constraint so we solve (4.44) first. The solutions are given in equations (4.51) and (4.52).

Before we solve (4.44) we note that  $\partial_{M_i} \rho^N = 0$  for  $i=q+1$  to  $R$  for our choice of  $S$ . We can write the constraint in a simpler form as

$$\begin{aligned} 0 = & - \sum_{i=1}^q \partial_{M_i} (\rho^{\bar{a}} \xi_{\bar{a}}^N) \sigma^{\bar{b}} W_{\bar{b} N M_1 \dots \hat{M}_i \dots M_R} - \rho^{\bar{a}} \xi_{\bar{a}}^N \partial_N (\sigma^{\bar{b}} W_{\bar{b} M_1 \dots M_R}) \\ & + \sum_{i=1}^q \partial_{M_i} (\sigma^{\bar{b}} \xi_{\bar{b}}^N) \rho^{\bar{a}} W_{\bar{a} N M_1 \dots \hat{M}_i \dots M_R} + \sigma^{\bar{b}} \xi_{\bar{b}}^N \partial_N (\rho^{\bar{a}} W_{\bar{a} M_1 \dots M_R}) \\ & - \rho^{\bar{a}} \sigma^{\bar{b}} C_{\bar{b} \bar{a}}^{\bar{c}} W_{\bar{c} M_1 \dots M_R} \end{aligned} \quad (4.45)$$

In solving this we need to consider three cases

(i)  $(\rho^{\bar{a}}, \sigma^{\bar{b}}) = (\epsilon^r, \eta^s)$ , (ii)  $(\epsilon^r, \tau^{mn}/2)$  and (iii)  $(\tau^{mn}/2, \tau^{kl}/2)$



i) Since  $C_{rs}^{\bar{c}} = 0$  and  $\partial_{M_i} \xi_r^N = 0$  the constraint reduces to

$$0 = \partial_r^W s_{M_1 \dots M_R} + \partial_s^W r_{M_1 \dots M_R}$$

The solution to this is

$$s_{M_1 \dots M_R}^W = \partial_s^F F_{M_1 \dots M_R} \quad (4.46)$$

$$\text{with } F_{M_1 \dots M_R}(x, \theta=0) = 0$$

ii) After substitution for the Killing vectors and factoring out  $\epsilon^{\tau mn}$  we obtain

$$\begin{aligned} 0 = & -\partial_{r mn} W_{M_1 \dots M_R} + \sum_{i=1}^q (\eta_{m M_i} W_{r n M_1 \dots \hat{M}_i \dots M_R} + \eta_{n M_i} W_{r m M_1 \dots \hat{M}_i \dots M_R}) \\ & + (\theta_m \partial_n + \theta_n \partial_m) W_{r M_1 \dots M_R} + \eta_{nr} W_{m M_1 \dots M_R} + \eta_{mr} W_{n M_1 \dots M_R} \end{aligned} \quad (4.47)$$

We rewrite this, using (4.11) and (4.13), as

$$\partial_r^W W_{mn M_1 \dots M_R} = \partial_r^{\Sigma'} W_{mn M_1 \dots M_R}$$

Using (4.46), the solution for  $W_{r M_1 \dots M_R}$ , this gives

$$\begin{aligned} W_{mn M_1 \dots M_R} &= A_{mn M_1 \dots M_R} + \Sigma'_{mn} F_{M_1 \dots M_R} \\ \text{where } A_{mn M_1 \dots M_R} &= W_{mn M_1 \dots M_R}(x, \theta=0) \end{aligned} \quad (4.48)$$

iii) For the third case the constraint is

$$\begin{aligned} 0 = & -\sum_{i=1}^q (\eta_{m M_i} W_{k l n M_1 \dots \hat{M}_i \dots M_R} + \eta_{n M_i} W_{k l m M_1 \dots \hat{M}_i \dots M_R}) - (\theta_m \partial_n + \theta_n \partial_m) W_{k l M_1 \dots M_R} \\ & + \sum_{i=1}^q (\eta_{k M_i} W_{m n l M_1 \dots \hat{M}_i \dots M_R} + \eta_{l M_i} W_{m n k M_1 \dots \hat{M}_i \dots M_R}) + (\theta_k \partial_l + \theta_l \partial_k) W_{mn M_1 \dots M_R} \\ & - \eta_{nk} W_{m l M_1 \dots M_R} - \eta_{ml} W_{k n M_1 \dots M_R} - \eta_{mk} W_{n l M_1 \dots M_R} - \eta_{nl} W_{k m M_1 \dots M_R} \end{aligned}$$

which we rewrite as

$$\Sigma'_{kl} W_{mn M_1 \dots M_R} - \Sigma'_{mn} W_{k l M_1 \dots M_R} = \eta_{km} W_{l n M_1 \dots M_R} + \eta_{kn} W_{m l M_1 \dots M_R} + \eta_{lm} W_{k n M_1 \dots M_R} + \eta_{ln} W_{m k M_1 \dots M_R}$$

Then using the solution (4.48) for  $W_{mnM_1 \dots M_R}$  we can write this in two parts as

$$\begin{aligned} \Sigma_{kl} A_{mnM_1 \dots M_R} - \Sigma_{mn} A_{klM_1 \dots M_R} = & \eta_{km} A_{lnM_1 \dots M_R} + \eta_{kn} A_{mlM_1 \dots M_R} + \eta_{lm} A_{knM_1 \dots M_R} \\ & + \eta_{ln} A_{mkM_1 \dots M_R} \end{aligned} \quad (4.49)$$

$$[\Sigma'_{kl}, \Sigma'_{mn}] F_{M_1 \dots M_R} = (\eta_{km} \Sigma'_{ln} + \eta_{kn} \Sigma'_{ml} + \eta_{lm} \Sigma'_{kn} + \eta_{ln} \Sigma'_{mk}) F_{M_1 \dots M_R}$$

The latter equation is an identity by (4.12).

The solution of the constraint (4.49) is calculated in Appendix D

We decompose  $A_{klM_1 \dots M_R}$  into

$$A_{klM_1 \dots M_R} = \alpha_1 A^{<q+2>}_{klM_1 \dots M_R} + \alpha_2 A^{<q>}_{klM_1 \dots M_R} + \alpha_3 A^{<q-2>}_{klM_1 \dots M_R}$$

where

$$\begin{aligned} A^{<q+2>}_{klM_1 \dots M_R} = & A_{klM_1 \dots M_R} + \sum_{i=1}^q (A_{kM_i lM_1 \dots \hat{M}_i \dots M_R} + A_{M_i l kM_1 \dots \hat{M}_i \dots M_R}) \\ & + \sum_{1 \leq i < j \leq q} A_{M_i M_j klM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R} \end{aligned}$$

$$A^{<q>}_{klM_1 \dots M_R} = \Sigma_{kl} \eta^{rs} \sum_{i=1}^q A_{rM_i sM_1 \dots \hat{M}_i \dots M_R}$$

$$A^{<q-2>}_{klM_1 \dots M_R} = \sum_{i=1}^q \sum_{j \neq i}^q \eta_{kM_i} \eta_{lM_j} A^{rs}_{rsM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}$$

and

$$\alpha_1 = 2/(q+1)(q+2) \quad \alpha_2 = -1/q(q+2) \quad \alpha_3 = 1/q(q+1) \quad (4.50)$$

Then we find that the only solutions to (4.49) are

$$A^{<q>}_{klM_1 \dots M_R} \quad \text{and} \quad A^{<q-2>=<0>}_{klM_1 \dots M_R}$$

Performing a  $\theta$  expansion of  $F_{M_1 \dots M_R}$

$$F_{M_1 \dots M_R} = \theta^S A_{SM_1 \dots M_R} + \frac{1}{2} \theta^2 \lambda_{M_1 \dots M_R}$$

We can write (4.46) as

$$W_{SM_1 \dots M_R} = A_{SM_1 \dots M_R} + \theta_S \lambda_{M_1 \dots M_R} \quad (4.51)$$

and (4.48) as

$$W_{mnM_1 \dots M_R} = A_{mnM_1 \dots M_R} + \theta^r \Sigma_{mn} A_{rM_1 \dots M_R} + \frac{1}{2} \theta^2 \Sigma_{mn} \lambda_{M_1 \dots M_R} \quad (4.52)$$

where  $A_{mnM_1 \dots M_R}$  is constrained by (4.49).

Now that we have the solutions (4.52) and (4.51) to the gauge parameter constraint we may use these to solve the field constraint (4.40). The solutions are given in equation (4.58).

This time we have only two cases to consider (i)  $\rho^{\bar{a}} = \epsilon^r$  and (ii)  $\tau^{mn}$ . Before we begin it is worth noting that since  $\partial_{M_i} \xi_a^N = 0$

for  $i=q+1$  to  $R$  the constraint may be written as

$$\begin{aligned} & -\sum_{i=1}^q \partial_{M_i} (\rho^{\bar{a}} \xi_a^N) V_{NM_1 \dots \hat{M}_i \dots M_R} - \rho^{\bar{a}} \xi_a^N \partial_N V_{M_1 \dots M_R} \\ & = \sum_{i=1}^q \partial_{M_i} (\rho^{\bar{a}} W_{\bar{a}M_1 \dots \hat{M}_i \dots M_R}) + \sum_{i=q+1}^R S(i) \partial_{M_i} (\rho^{\bar{a}} W_{\bar{a}M_1 \dots \hat{M}_i \dots M_R}) \end{aligned}$$

i) Since  $\partial_{M_i} \xi_a^N = 0$  for  $i=1$  to  $q$  the constraint becomes

$$-\partial_r V_{M_1 \dots M_R} = -\sum_{i=1}^q \partial_{M_i} W_{rM_1 \dots \hat{M}_i \dots M_R} + \sum_{i=q+1}^R S(i) \partial_{M_i} W_{rM_1 \dots \hat{M}_i \dots M_R}$$

From (4.51) this gives

$$\partial_r V_{M_1 \dots M_R} = \sum_{i=1}^q \eta_{rM_i} \lambda_{M_1 \dots \hat{M}_i \dots M_R} - \sum_{i=q+1}^R S(i) \partial_{M_i} (A_{rM_1 \dots \hat{M}_i \dots M_R} + \theta^r \lambda_{M_1 \dots \hat{M}_i \dots M_R})$$

Hence

$$\begin{aligned} V_{M_1 \dots M_R} &= V_{M_1 \dots M_R} + \sum_{i=1}^q \theta^r \eta_{rM_i} \lambda_{M_1 \dots \hat{M}_i \dots M_R} \\ &\quad - \sum_{i=q+1}^R S(i) \partial_{M_i} (\theta^r A_{rM_1 \dots \hat{M}_i \dots M_R} + \frac{1}{2} \theta^2 \lambda_{M_1 \dots \hat{M}_i \dots M_R}) \end{aligned} \quad (4.53)$$

ii) The constraint is

$$\begin{aligned}
 & - \sum_{i=1}^q (\eta_{mM_i} V_{nM_1 \dots \hat{M}_i \dots M_R} + \eta_{nM_i} V_{mM_1 \dots \hat{M}_i \dots M_R}) - (\theta_m \partial_n + \theta_n \partial_m) V_{M_1 \dots M_R} \\
 & = \sum_{i=1}^q \partial_{M_i} W_{mnM_1 \dots \hat{M}_i \dots M_R} + \sum_{i=q+1}^R S(i) \partial_{M_i} W_{mnM_1 \dots \hat{M}_i \dots M_R}
 \end{aligned}$$

If we substitute the solutions (4.52) and (4.53) then to zeroth order in  $\theta$  we obtain

$$\begin{aligned}
 & - \sum_{i=1}^q (\eta_{mM_i} V_{nM_1 \dots \hat{M}_i \dots M_R} + \eta_{nM_i} V_{mM_1 \dots \hat{M}_i \dots M_R}) \\
 & = \sum_{i=1}^q \Sigma_{mn} A_{M_i M_1 \dots \hat{M}_i \dots M_R} + \sum_{i=q+1}^R S(i) \partial_{M_i} A_{mnM_1 \dots \hat{M}_i \dots M_R}
 \end{aligned}$$

We can write this as

$$\Sigma_{mn} V_{M_1 \dots M_R} = - \sum_{i=1}^q \Sigma_{mn} A_{M_i M_1 \dots \hat{M}_i \dots M_R} = - \sum_{j=q+1}^R S(j) \partial_{M_j} A_{mnM_1 \dots \hat{M}_j \dots M_R}$$

If  $q=0$  this does not place any constraint on  $V_{M_1 \dots M_R}$ . If  $q \neq 0$  then we can operate on both sides with  $\Sigma^{mn}$ . From Appendix C equation (C.1) and the symmetry properties of  $V_{M_1 \dots M_R}$  (4.36) we obtain

$$-2q(q+2) V_{M_1 \dots M_R} = 2q(q+2) \sum_{i=1}^q A_{M_i M_1 \dots \hat{M}_i \dots M_R} - \Sigma^{mn} \sum_{j=q+1}^R S(j) \partial_{M_j} A_{mnM_1 \dots \hat{M}_j \dots M_R}$$

Now  $A_{mnM_1 \dots M_R}$  is constrained by (4.49) with solution

$$A_{mnM_1 \dots M_R} = -A^{<q>}_{mnM_1 \dots M_R} / q(q+2) + \beta A^{<q-2>}_{mnM_1 \dots M_R} / q(q+1)$$

where  $\beta=1$  for  $q=2$  and is zero otherwise. From (4.50) we have

$$\Sigma^{mn} A^{<q>}_{mnM_1 \dots M_R} = -2q(q+2) \sum_{i=1}^q A_{rM_i M_1 \dots \hat{M}_i \dots M_R}$$

$$\Sigma^{mn} A^{<q-2>}_{mnM_1 \dots M_R} = 0$$

Then our constraint on  $V_{M_1 \dots M_R}$  for  $q \neq 0$  becomes

$$V_{M_1 \dots M_R} = - \sum_{i=1}^q A_{M_i M_1 \dots \hat{M}_i \dots M_R} + \sum_{j=q+1}^R S(j) \partial_{M_j} \sum_{i=1}^q A_{rM_i M_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R} / q(q+2) \quad (4.54)$$

We now have all the information we need to write the field solutions. Before we do so it is convenient to perform some field redefinitions.

We redefine  $A_{M_1 M_2 \dots \hat{M}_i \dots M_R}$  in (4.54) for  $i=1$  to  $q$  as

$$A'_{M_1 M_2 \dots \hat{M}_i \dots M_R} = A_{M_1 M_2 \dots \hat{M}_i \dots M_R} - \sum_{j=q+1}^R S(j) \partial_{M_j} A_{r M_1 \dots \hat{M}_i \dots M_j \dots M_R} / q(q+2) \quad (4.55)$$

then dropping the prime we have

$$v_{M_1 \dots M_R} = - \sum_{i=1}^q A_{M_1 \dots M_i \dots \hat{M}_i \dots M_R}$$

The solution of the field constraint, from (4.53), is then

$$\begin{aligned} v_{M_1 \dots M_R} = & x_{M_1 \dots M_R} - \sum_{i=1}^q (A_{M_1 M_2 \dots \hat{M}_i \dots M_R} + \theta_{M_i} \lambda_{M_1 \dots \hat{M}_i \dots M_R}) \\ & - \sum_{i=q+1}^R S(i) \partial_{M_i} (\theta^r A_{r M_1 \dots \hat{M}_i \dots M_R} + \frac{1}{2} \theta^2 \lambda_{M_1 \dots \hat{M}_i \dots M_R}) \end{aligned} \quad (4.56)$$

where  $x_{M_1 \dots M_R} \neq 0$  only if  $q = 0$ .

Note the redefinition (4.55) has left the term

$$- \sum_{i=q+1}^R S(i) \partial_{M_i} (\theta^r A_{r M_1 \dots \hat{M}_i \dots M_R})$$

unchanged since it is a generalised curl.

There is one further redefinition we would like to make.  $A_{m M_1 \dots M_R}$  has the symmetry property

$$A_{m M_1 \dots M_i \dots M_j \dots M_R} = A_{m M_1 \dots M_j \dots M_i \dots M_R} \quad i, j=1 \text{ to } q$$

Hence we can construct only  $q+1$  terms from  $A_{m M_1 \dots M_R}$  by interchange of  $Sp(2)$  indices. These are  $A_{m M_1 \dots M_R}$  and  $A_{M_i m M_1 \dots \hat{M}_i \dots M_R}$  for  $i=1$  to  $q$ . Now

we can decompose  $A_{m M_1 \dots M_R}$  into

$$A_{m M_1 \dots M_R} = \beta^{<q+1>} A^{<q+1>}_{m M_1 \dots M_R} + \beta^{<q-1>} A^{<q-1>}_{m M_1 \dots M_R}$$

where

$$A^{<q+1>}_{mM_1 \dots M_R} = \sum_{i=1}^q A_{M_i m M_1 \dots \hat{M}_i \dots M_R} + A_{m M_1 \dots M_R}$$

$$A^{<q-1>}_{mM_1 \dots M_R} = \sum_{i=1}^q n_{m M_i} A^r_{M_1 \dots \hat{M}_i \dots M_R} = q A_{m M_1 \dots M_R} - \sum_{i=1}^q A_{M_i m M_1 \dots \hat{M}_i \dots M_R}$$

$$\text{Hence } \beta^{<q+1>} = \beta^{<q-1>} = 1/(1+q) .$$

In (4.56) we redefine  $\lambda_{M_1 \dots \hat{M}_i \dots M_R}$  for  $i=1$  to  $q$  by

$$\lambda'_{M_1 \dots \hat{M}_i \dots M_R} = \lambda_{M_1 \dots \hat{M}_i \dots M_R} - \sum_{j=q+1}^R S(j) \partial_{M_j} A^r_{M_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R} \quad (4.57)$$

Then denoting  $A^{<q+1>}_{mM_1 \dots M_R}$  by  $A_{(mM_1 \dots M_q) \dots M_R}$  we write our final

solution for  $V_{M_1 \dots M_R}$  as

$$\begin{aligned} V_{M_1 \dots M_R} &= X_{M_1 \dots M_R} - A_{(M_1 \dots M_q) \dots M_R} - \sum_{i=1}^q \theta_{M_i} \lambda_{M_1 \dots \hat{M}_i \dots M_R} \\ &- \sum_{i=q+1}^R S(i) \partial_{M_i} (\theta^r A_{(rM_1 \dots M_q) \dots \hat{M}_i \dots M_R} / (q+1) + \frac{1}{2} \theta^2 \lambda_{M_1 \dots \hat{M}_i \dots M_R}) \end{aligned} \quad (4.58)$$

where we have dropped the primes,  $X_{M_1 \dots M_R} \neq 0$  only if  $q = 0$  and the term

$$- \sum_{i=q+1}^R S(i) \partial_{M_i} (\frac{1}{2} \theta^2 \lambda_{M_1 \dots \hat{M}_i \dots M_R})$$

is a generalised curl hence unaffected by the redefinition (4.57).

Having obtained our solution (4.58) to the field constraint we now compare our results with those of other authors. Townsend [6] was the first to attempt to covariantly quantise the rank-two antisymmetric field. He applied a naive Fadeev-Popov technique and obtained two, vector, fermionic ghosts and four, scalar, bosonic ghosts. However this was in disagreement with the requirement that the theory should be unitary. As it stood the theory had two degrees of freedom instead of the one degree required. Townsend then had to appeal to Ward identities to decouple the extra degree of freedom. Namazie and Storey [7] then

extended this technique to the rank-three case. They obtained two second-rank fermionic ghosts, four vector bosonic ghosts and eight scalar fermionic ghosts. This was in apparent agreement with unitarity and the requirement that the rank-three field be non-propagating.

Our own results are contained in (4.58). The fields  $A_{M_1 \dots M_R}$  are the ghost fields which accompany the field  $V_{\mu_1 \dots \mu_2}$  where the  $\mu_i$  are even indices. For  $q$  odd  $A_{M_1 \dots M_R}$  is fermionic and for  $q$  even it is bosonic. The ghost field has the same symmetry properties as  $V_{M_1 \dots M_R}$  given in (4.36). Hence a rank- $R$  antisymmetric tensor is accompanied by two rank  $(R-1)$  fermionic ghosts, three rank  $(R-2)$  bosonic ghosts, ..., down to  $R+1$  scalar ghosts. A simple degree-of-freedom counting argument [8] gives

$$\sum_{q=0}^R (-1)^q (q+1) \binom{4}{R-q} \quad (4.59)$$

degrees of freedom. Note the  $\lambda$  are auxiliary fields and do not contribute to the count. In particular for the second-rank field we obtain two vector fermionic ghosts, only three scalar bosonic ghosts and the required one degree of freedom. For the third-rank case we find two second-rank fermionic ghosts, three vector bosonic ghosts, four scalar fermionic ghosts and no degrees of freedom as required.

Our results are in agreement with the remaining authors [5,9-13]. The ghost spectrum given above was derived to formally ensure unitarity of the theory by Siegel [9] in a path integral approach, and by Thierry-Mieg and Baulieu [10]. The degree of freedom count (4.50) has been given by these two authors and by Hata et al [5]. The rank two and three cases have been derived explicitly in a superspace approach to BRST symmetry by Marchetti and Tonin [11]. A superspace approach to the rank- $R$  case was given by Kawasaki and Kimura [12]. The unitarity

of theories in agreement with our ghost-spectrum and degree-of freedom count has been proved by the 'quartet mechanism' of Kugo and Ojima [14]. For example, in the rank-two case see Kimura [13]. We have nothing to add to these proofs.

Having demonstrated that the CSDR approach to covariant quantisation of antisymmetric rank-R fields is in agreement with the results of other techniques we consider in the next section the choice of an action and its reduction to four dimensions.

#### 4.4 BRST INVARIANT ACTION FOR THE KALB-RAMOND FIELD

In this section we consider the choice of a BRST invariant action in six dimensions for the Kalb-Ramond field and the reduction to the four dimensional theory. The extension of the method to higher rank fields is clear. We use our solution (4.23) to the field constraint rather than the  $R=2$  case of (4.58) since the notation is a little clearer.

In four dimensions the Kalb-Ramond action is given by

$$\int d^4X L_{KR} = \int d^4X \frac{1}{2} v^{\mu\nu} \partial_\sigma \partial_\sigma v_{\mu\nu} + \partial^\mu v_{\mu\nu} \partial_\sigma v^{\sigma\nu} \quad (4.60)$$

A natural extension of this to six dimensions is given by

$$\int d^6X X^2 L_{KR} = \int d^6X X^2 \left( -\frac{1}{2} V^{KL} \partial_M \partial_M V_{LK} + \partial^K V_K^L \partial^M V_{ML} \right) \quad (4.61)$$

Now this action is invariant under gauge transformations

$$V_{MN} \rightarrow V_{MN} + \partial_M T_N - [MN] \partial_N T_M$$

for some gauge parameter  $T_N$ . From our solution to the field constraint (4.34) we can clearly evaluate (4.51) in a gauge in which only  $v_{\mu\nu}$  survives. Then (4.51) immediately reduces to the required four dimensional form after integration over  $\theta$ .



Having chosen a gauge in which only  $v_{\mu\nu}$  survives there is still a degree of gauge freedom left. To remove this we add to the action, a gauge fixing term

$$S = \int d^6X \, X^2 L_{KR} + L_{GF}$$

$$\int d^6X \, L_{GF} = \int d^6X \, v^{KL} \alpha_L^N v_{NK/\mu} \quad (4.62)$$

and in particular choose  $\alpha_L^N = \delta_L^N$ .

We expand the gauge fixing term

$$\int d^6X \, L_{GF} = \int d^6X \, -v^{\mu\nu} v_{\mu\nu} - 2v^{\mu 1} v_{\mu 1} + v^1 q_{1q}$$

Then from (4.35) we have for the first term

$$\begin{aligned} \int d^6X \, -v^{\mu\nu} v_{\mu\nu} &= \int d^6X \, -(v^{\mu\nu} + \theta^m (\partial^\nu c_m^\mu - \partial^\mu c_m^\nu) + \frac{1}{2}\theta^2 (\partial^\nu \lambda^\mu - \partial^\mu \lambda^\nu)) \\ &\quad \times (v_{\mu\nu} + \theta^n (\partial_\nu c_{\mu n} - \partial_\mu c_{\nu n}) + \frac{1}{2}\theta^2 (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu)) \\ &= \int d^6X \, -v^{\mu\nu} (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) - (\partial^\nu \lambda^\mu - \partial^\mu \lambda^\nu) v_{\mu\nu} + (\partial^\nu c^{\mu n} - \partial^\mu c^{\nu n}) (\partial_\nu c_{\mu n} - \partial_\mu c_{\nu n}) \\ &= \int d^6X \, 2\lambda^\mu \partial^\nu v_{\mu\nu} - 2v^{\mu\nu} \partial_\nu \lambda_\mu + 2c^{\mu n} \partial^\nu \partial_\mu c_{\nu n} - 2c^{\mu n} \partial^\nu \partial_\nu c_{\mu n} \end{aligned}$$

where we have used  $\theta^m \theta^n = \frac{1}{2}\theta^2 \epsilon^{nm}$  to obtain the second line. The second term gives

$$\begin{aligned} \int d^6X \, -2v^{\mu 1} v_{\mu 1} &= \int d^6X \, -2(c^{\mu 1} + \theta^1 \lambda^\mu + \frac{1}{2}\theta^m \partial^\mu c_m^1 - \frac{1}{2}\theta^2 \partial^\mu \lambda^1) \\ &\quad \times (c_{\mu 1} + \theta_1 \lambda_\mu + \frac{1}{2}\theta^m \partial_\mu c_{1m} - \frac{1}{2}\theta^2 \partial_\mu \lambda_1) \\ &= \int d^6X \, 2c^{\mu 1} \partial_\mu \lambda_1 - 2\lambda^1 \partial^\mu c_{\mu 1} - 4\lambda^\mu \lambda_\mu + \frac{1}{2}c^1 \partial^\mu \partial_\mu c_{1n} \end{aligned}$$

since  $c_{1q} = c_{q1}$  implies  $c^1_1 = 0$ . For the third term we obtain

$$\begin{aligned} \int d^6X \, v^1 q_{1q} &= \int d^6X \, (c^1 q - \theta^1 \lambda^q - \theta^q \lambda^1) (c_{1q} - \theta_1 \lambda_q - \theta_q \lambda_1) \\ &= \int d^6X \, -6\lambda^q \lambda_q \end{aligned}$$

The action for the Kalb-Ramond field is then

$$S = \int d^4x \left[ \frac{1}{2} v^{\mu\nu} \partial_\sigma \partial_\nu v_\mu + \partial_\mu v_\nu \partial_\sigma v^{\sigma\nu} + (2\lambda^\mu \partial_\nu v_{\mu\nu} - 2v^{\mu\nu} \partial_\nu \lambda_\mu + 2c^{\mu\nu} \partial_\mu c_{\nu\lambda} - 2c^{\mu\lambda} \partial_\nu c_{\nu\mu} + 2c^{\mu\lambda} \partial_\mu c_{\nu\lambda} - 2\lambda^\mu \partial_\mu c_{\nu\lambda} - 4\lambda^\mu \lambda_\mu + \frac{1}{2} c^{\mu\lambda} \partial_\mu c_{\nu\lambda} - 6\lambda^\mu \lambda_\mu) / \mu \right] \quad (4.63)$$

We can easily check that this action is invariant under extended BRST transformations. Under  $\theta^m \rightarrow \theta^m + \rho^m$  our field constraint solutions (4.35) transform as

$$\delta v_{\mu\nu} = \rho^m (\partial_\nu c_{\mu m} - \partial_\mu c_{\nu m}) + (\rho^5 \theta^6 + \theta^5 \rho^6) (\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu)$$

$$\delta v_{\mu 1} = \rho^m (\eta_{1m} \lambda_\mu + \frac{1}{2} \partial_\mu c_{1m}) - (\rho^5 \theta^6 + \theta^5 \rho^6) \partial_\mu \lambda_1$$

$$\delta v_{1q} = \rho^m (\eta_{m1} \lambda_q + \eta_{mq} \lambda_1)$$

Now if we define the variations of the components of the fields by

$$\delta v_{\mu\nu} = \delta v_{\mu\nu} + \theta^m (\partial_\nu \delta c_{\mu m} - \partial_\mu \delta c_{\nu m}) + \frac{1}{2} \theta^2 (\partial_\nu \delta \lambda_\mu - \partial_\mu \delta \lambda_\nu)$$

$$\delta v_{\mu 1} = \delta c_{\mu 1} + \theta^m (\eta_{1m} \delta \lambda_\mu + \frac{1}{2} \partial_\mu \delta c_{1m}) - \frac{1}{2} \theta^2 \partial_\mu \delta \lambda_1$$

$$\delta v_{1q} = \delta c_{1q} + \theta^m (\eta_{m1} \delta \lambda_q + \eta_{mq} \delta \lambda_1)$$

then by comparison of these two sets of equations we obtain

$$\delta v_{\mu\nu} = \rho^m (\partial_\nu c_{\mu m} - \partial_\mu c_{\nu m})$$

$$\delta c_{\mu 1} = \rho^m (\eta_{1m} \lambda_\mu + \frac{1}{2} \partial_\mu c_{1m})$$

$$\delta c_{1q} = \rho^m (\eta_{m1} \lambda_q + \epsilon_{mq} \lambda_1) \quad (4.64)$$

$$\delta \lambda_1 = 0$$

$$\delta \lambda_\mu = \frac{1}{2} \rho^m \partial_\mu \lambda_m$$

These are the variations of the fields under an extended BRST transformation. Substituting these expressions into the variation of the action (4.63) it is easily found that  $S$  is invariant.

Now the  $\lambda_k$  are simply auxiliary fields and can be eliminated through their equations of motion. For a field  $\phi_M$  and Lagrangian  $L$  the equation of motion is

$$0 = \frac{\partial L}{\partial \phi_M} - [MN] \frac{\partial^N L}{\partial \phi_M^N}$$

Thus for  $\lambda_\mu$  and  $\lambda_l$  we obtain

$$0 = \partial_\nu v^{\mu\nu} - 2\lambda^\mu$$

$$0 = 3\lambda^l + \partial_\mu c^{\mu l}$$

respectively. Note that these equations are consistent with the variations given in (4.64). Eliminating  $\lambda_N$  from the action we obtain

$$S = \int d^4x \left[ \frac{1}{2} v^{\mu\nu} \partial^\sigma \partial_\sigma v_{\mu\nu} + (1+\mu) \partial^\mu v_{\mu\nu} \partial^\nu v^{\sigma\nu} / \mu \right. \\ \left. + (-2c^{l\mu} \partial^\sigma \partial_\sigma c_{l\mu} + 4c^{l\mu} \partial_\nu \partial_\mu c_{l\nu} / 3 + \frac{1}{2} c^{ln} \partial^\sigma \partial_\sigma c_{ln}) / \mu \right] \quad (4.65)$$

In this expression the results discussed at the end of Section 4.3 are made explicit. We have covariantly quantised the second-rank anti-symmetric field and find two vector fermionic ghosts  $c^{l\mu}$  for  $l = 1, 2$  and three scalar bosonic ghosts  $c^{ln} = c^{nl}$  i.e.  $c^{11}$ ,  $c^{12}$  and  $c^{22}$ . The gauge fixing term for the Kalb-Ramond field, the associated vector ghosts, the gauge fixing term for the vector ghosts and the associated secondary ghosts have all arisen automatically from the dimensional reduction procedure in the correct form to ensure unitarity of the theory. Note that a different choice of  $\alpha_L^N$  in (4.52) would simply lead to a different coefficient for the ghost gauge fixing term after appropriate redefinitions of the fields.

Finally, we compare our action (4.55) with those of other authors. Our correspondence with the results of Kimura [13] may be seen if we identify our third bosonic ghost  $c_{12}$  as a linear combination of their

$B_{03}$  and  $B_3$ . (4.65) corresponds to (14) in the work of Marchetti and Tonin [11] provided one replaces, correctly, their second last term  $\partial^\mu \xi B_\mu$  by  $-\partial^\mu \xi \partial_\mu \xi/4$ .

This completes our chapter on the BRST quantisation of higher rank antisymmetric fields. In the next chapter we apply the same method to another type of field.

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## 5 BRST QUANTISATION OF THE MASSLESS RARITA-SCHWINGER FIELD

### 5.1 PREAMBLE

The superpartner of the graviton is a spin-3/2 vector gauge field, the Rarita-Schwinger field or 'gravitino'. The covariant quantisation of massless Rarita-Schwinger fields has seen various treatments [1-5]. Early work on supergravity established by perturbative arguments the existence of non-standard quartic ghost terms [2]. This was confirmed by analysis of the correct auxiliary fields [3], and from canonical [4] and BRST arguments [5]. At the same time degree-of-freedom decoupling [6] required together with the bosonic Fadeev-Popov spinor ghosts, a further fermionic, Nielsen-Kallosh spinor ghost [7]. Subsequent BRST arguments also supported this [8,9].

In Chapter 2 we discussed the successful application of CSDR to BRST quantisation of vector gauge fields. The fields were taken to be in a vector representation of  $OSp(4/2)$  and dimensional reduction was performed over the coset space  $Sp(2) \wedge T_2 / Sp(2)$ . In Chapter 4 we discussed the successful extension of this technique to higher-rank, antisymmetric-tensor, gauge fields. The superspace approach to BRST quantisation has not previously been applied to spinor-vector gauge fields. Indeed degree-of-freedom counting, in this context simply the superdimension of the relevant gauge field, seems to fail as finite-dimensional spinor representations of  $OSp(4/2)$  have zero superdimension [10]. However infinite-dimensional spinors do exist with non-zero superdimension [11].

In this chapter we extend the CSDR method of BRST quantisation to spinor-vector gauge fields in an infinite dimensional representation

of  $OSp(4/2)$ . Our work is based on the material contained in reference [12]. Dimensional reduction over the coset-space  $Sp(2) \wedge T_2 / Sp(2)$  ensures that the final action is BRST invariant with the BRST transformations generated by supertranslations of the constraint solutions.

In Section 5.2 we set up the constraint equations for the fields and the gauge parameters. We then solve these equations in Section 5.3 and in Section 5.4 use the solutions to find the dimensionally reduced theory. We conclude the chapter, in Section 5.5, with a discussion of our results.

## 5.2. CONSTRAINT EQUATIONS

The constraint equations for fermions have been derived by Manton [13] for a general gauge group  $G$  and coset-space  $S/R$ . We briefly reproduce his results here.

Suppose the infinitesimal action of  $S$  with generators  $J_{\bar{a}}$ , on the coordinates is

$$X^M \rightarrow X^M + \rho^M = X^M + \rho^{\bar{a}} \xi_{\bar{a}}^M$$

In treating spinor fields it is necessary to introduce explicitly the vielbein  $E_M^A$  which is taken to be a suitable background on  $M^4$  and the invariant canonical form on  $S/R$ . The corresponding metric

$$g_{MN} = \eta_{AB} E_M^A E_N^B \quad [AN]$$

is form invariant,  $L_\rho g_{MN} = 0$ . However the vielbein need only be invariant up to a local Lorentz transformation

$$L_\rho E_M^A = E_M^B \Lambda_B^A$$

In consequence spinor fields must transform, contragradiently as

$$L_\rho \Psi = -\Lambda^\rho \Psi \quad (5.1)$$

in the appropriate spinor representation of the Lorentz group.

Then the massless spinor-vector fields are said to be S-invariant if the action of S given by

$$L_{\rho} \Psi_M = -\rho^L \partial_L \Psi_M - (\partial_M \rho^L) \Psi_L \quad (5.2)$$

can be compensated by a gauge transformation,  $\chi^{\rho} = \rho^{\bar{a}} \chi_{\bar{a}}$ , in addition to the induced Lorentz transformation. The field constraint is thus

$$L_{\rho} \Psi_M = \nabla_M \chi^{\rho} - \Lambda^{\rho} \Psi_M \quad (5.3)$$

where  $\nabla_M = \partial_M + \Omega_M$  and  $\Omega_M$  is the torsion-free, spin connection. The spin-connection [14] is introduced to ensure that  $\nabla_M \Psi$  transforms as a vector hence

$$L_{\rho} \Omega_M = \nabla_M \Lambda^{\rho} \quad (5.4)$$

from (5.3).

Now the commutator of two Lie derivatives is another one

$$[L_{\rho}, L_{\sigma}] = L_{[\rho, \sigma]} \quad (5.5)$$

where  $[\rho, \sigma]^{\bar{a}} = -\rho^{\bar{b}} \sigma^{\bar{c}} C_{\bar{b}\bar{c}}^{\bar{a}}$  and  $C_{\bar{b}\bar{c}}^{\bar{a}}$  are the structure constants of S.

Applying this to (5.4) we obtain

$$L_{\rho} \Lambda^{\sigma} - L_{\sigma} \Lambda^{\rho} + [\Lambda^{\rho}, \Lambda^{\sigma}] - \Lambda^{[\rho, \sigma]} = 0 \quad (5.6)$$

The method is the same as was used in Chapter 4 to derive (4.4).

Similarly applying (5.5) to (5.3) we obtain the gauge parameter constraint

$$L_{\rho} \chi^{\sigma} - L_{\sigma} \chi^{\rho} + \Lambda^{\rho} \chi^{\sigma} - \Lambda^{\sigma} \chi^{\rho} - \chi^{[\rho, \sigma]} = 0 \quad (5.7)$$

In deriving this we need to use (5.4) and (5.6).



### 5.3 CONSTRAINT SOLUTIONS

Now in order to obtain the BRST quantisation of the spinor-vector fields we choose  $G = \text{OSp}(4/2)$  and  $S/R = \text{Sp}(2) \wedge T_2 / \text{Sp}(2)$  as discussed in Section 2.4. To find the embedding of  $R$  in  $G$  and  $R$  in  $S$  we need to consider representations of  $G$ .

Under an infinitesimal  $\text{OSp}(4/2)$  transformation we have

$$\delta X^M = X^N \lambda_N^M \quad (5.8)$$

If we define the generators of  $\text{OSp}(4/2)$  in the vector representation by

$$\delta X^M = X^N (\frac{1}{2} \lambda^{LK} J_{KL}^V)_N^M = X^N \frac{1}{2} \lambda^{LK} [LN][KN] (J_{KL}^V)_N^M$$

then (5.8) implies

$$(J_{KL}^V)_N^M = [LN][KN] (\delta_K^M \eta_{NL} - [KL] \delta_L^M \eta_{NK}) \quad (5.9)$$

Note this is consistent with the commutation relation

$$J_{PQ} J_{RS} - [PR][PS][QR][QS] J_{RS} J_{PQ} = -\eta_{RQ} J_{PS} + [PQ] \eta_{RP} J_{QS} \\ + [RS] \eta_{SQ} J_{PR} - [PQ][RS] \eta_{SP} J_{QR} \quad (5.10)$$

for the generators of  $\text{OSp}(4/2)$ .

In a spinor representation we take

$$J_{KL}^S = -(\Gamma_K \Gamma_L - \Gamma_L \Gamma_K [KL]) / 4 \quad (5.11)$$

Then provided the generalised Dirac matrices,  $\Gamma_K$ , obey

$$\Gamma_K \Gamma_L + [KL] \Gamma_L \Gamma_K = 2\eta_{LK} \quad (5.12)$$

$J_{KL}^S$  obeys (5.10). It will be convenient to define  $\Sigma_{KL} = J_{KL}^S$ .

Spinors,  $\Psi$ , then transform contragrediently, as

$$\delta \Psi = -\frac{1}{2} \lambda^{KL} \Sigma_{LK} \Psi \quad (5.13)$$

Also  $\Gamma_M \Psi$  transforms like a spinor-vector

$$\delta (\Gamma_M \Psi) = -\frac{1}{2} \lambda^{KL} \Sigma_{LK} \Gamma_M \Psi - \lambda_M^N \Gamma_N \Psi$$

as does  $\Psi_M$ , and  $\Gamma^M \Psi_M$  transforms as a spinor.

Now the embedding of  $R$  in  $G$  is fixed by identifying the generators of  $Sp(2)$  as  $\Sigma_{mn}$ . For the generators of  $S$  we take  $\Sigma_{mn}$  and  $P_q$  with commutation relations as in Section 2.4 which we repeat here for convenience

$$\begin{aligned} [\Sigma_{mn}, \Sigma_{pq}] &= \eta_{np} \Sigma_{mq} + \eta_{nq} \Sigma_{pm} + \eta_{mp} \Sigma_{nq} + \eta_{mq} \Sigma_{pn} \\ [\Sigma_{mn}, P_q] &= \eta_{mq} P_n - \eta_{nq} P_m \\ \{P_q, P_r\} &= 0 \end{aligned} \quad (5.14)$$

Hence in the field constraint (5.3) we have  $\Lambda^0 = \rho^{\bar{a}} \Lambda_{\bar{a}}$  with

$$\Lambda_r = 0, \Lambda_{mn} = \Sigma_{mn} \quad (5.15)$$

Also for this choice of coset-space we have  $E_M^A = \delta_M^A$  following Salam and Strathdee [14]. Hence our spin connection,  $\Omega_M$ , is zero.

The Killing vectors  $\xi_{\bar{a}}^M$  have been given also in Section 2.4. We have

$$\xi_k^m = \delta_k^m, \xi_{kl}^m = \theta_k \delta_l^m - \theta_l \delta_k^m \quad (5.16)$$

We now have sufficient information to solve the constraints (5.3) and (5.7). The method is the same as in the previous chapter. First we solve the gauge parameter constraint then we use these solutions in (5.3) to find the  $S$ -invariant fields.

In order to solve (5.7) for the gauge parameter we need to consider the cases (i)  $(\rho^{\bar{a}}, \sigma^{\bar{b}}) = (\epsilon^k, \epsilon^l)$ , (ii)  $(\frac{1}{2}\tau^{kl}, \epsilon^m)$  and (iii)  $(\frac{1}{2}\tau^{kl}, \frac{1}{2}\tau^{mn})$ .

i) Since  $C_{kl}^m = 0$  and  $\Lambda^\epsilon = 0$  the constraint becomes  $L_\rho \chi^\sigma - L_\sigma \chi^\rho = 0$ .

After substitution of the Killing vectors this becomes

$$\partial_k \chi_l + \partial_l \chi_k = 0$$

The most general solution of this is

$$\chi_l = \partial_l \Phi \quad (5.17)$$

for some  $\Phi$  whose  $\theta$  expansion has no terms independent of  $\theta$ .

ii) The constraint (5.7) becomes

$$\partial_m x_{kl} = -\Sigma_{kl} x_m + \theta_k \partial_l x_m + \theta_l \partial_k x_m + \eta_{km} x_l + \eta_{lm} x_k$$

Using (5.17) for  $x_l$  this becomes

$$\partial_m x_{kl} = -\partial_m (\Sigma_{kl} \phi - \theta_k \partial_l \phi - \theta_l \partial_k \phi)$$

The solution for  $x_{kl}$  is

$$x_{kl} = x_{kl}^{(0)} - \Sigma_{kl} \phi + \theta_k \partial_l \phi + \theta_l \partial_k \phi \quad (5.18)$$

$$\text{where } x_{kl}^{(0)} = x_{kl}(x, \theta=0)$$

iii) The constraint is

$$\begin{aligned} & -\theta_k \partial_l x_{mn} - \theta_l \partial_k x_{mn} + \theta_m \partial_n x_{kl} + \theta_n \partial_m x_{kl} + \Sigma_{kl} x_{mn} - \Sigma_{mn} x_{kl} \\ & = \eta_{km} x_{ln} + \eta_{kn} x_{ml} + \eta_{lm} x_{kn} + \eta_{ln} x_{mk} \end{aligned}$$

Substituting (5.18) we obtain at zeroth order in  $\theta$

$$\Sigma_{kl} x_{mn}^{(0)} - \Sigma_{mn} x_{kl}^{(0)} = \eta_{km} x_{ln}^{(0)} + \eta_{kn} x_{ml}^{(0)} + \eta_{lm} x_{kn}^{(0)} + \eta_{ln} x_{mk}^{(0)} \quad (5.19)$$

and an identity for the higher orders. (5.19) with (5.14)

implies

$$x_{mn}^{(0)} = \Sigma_{mn} \phi \quad (5.20)$$

for some  $\phi$ .

Performing a  $\theta$  expansion of  $\phi$

$$\phi = \theta^a \phi_a + \frac{1}{2} \theta^2 b$$

we can write the solutions (5.17) and (5.18) with (5.20) as

$$x_l = \phi_l + \theta_l b \quad (5.21)$$

$$x_{kl} = \Sigma_{kl} \phi + \theta^r (-\Sigma_{kl} \phi_r + \epsilon_{kr} \phi_l + \epsilon_{lr} \phi_k) - \frac{1}{2} \theta^2 \Sigma_{kl} b \quad (5.22)$$

Having obtained the most general solutions for the gauge parameters

$x_l$  and  $x_{kl}$  we substitute these solutions into the constraint (5.3) and solve for  $\Psi$ . Now there are four cases to be considered (i)  $(M, \rho^{\bar{a}}) = (\mu, \epsilon^k)$ , (ii)  $(\mu, \tau^{mn}/2)$ , (iii)  $(m, \epsilon^k)$  and (iv)  $(m, \tau^{kl}/2)$ . The values of  $\Lambda$  are given in (5.15) and the Killing vectors are given in (5.16).

i) Since  $\Omega_M = 0$  we have  $\nabla_M = \partial_M$  then the constraint on  $\psi$  becomes

$\partial_n \psi_\mu = -\partial_\mu \chi_n$ . Using (5.21) this gives

$$\partial_n \psi_\mu = -\partial_\mu (\phi_n + \theta_n b)$$

The solution to this is

$$\psi_\mu = \psi_\mu^{(0)} - \theta^n \partial_\mu \phi_n - \frac{1}{2} \theta^n \theta_n \partial_\mu b \quad (5.23)$$

ii) We obtain

$$-\theta_m \partial_n \psi_\mu - \theta_n \partial_m \psi_\mu = \partial_\mu \chi_{mn} - \Sigma_{mn} \psi_\mu$$

Substituting (5.22) and (5.23) into this we find

$$\Sigma_{mn} \psi_\mu^{(0)} = \Sigma_{mn} \partial_\mu \phi$$

This implies that our most general solution for  $\psi_\mu$  is

$$\psi_\mu = \psi'_\mu + \partial_\mu \phi - \theta^n \partial_\mu \phi_n - \frac{1}{2} \theta^n \theta_n \partial_\mu b \quad (5.24)$$

where  $\Sigma_{mn} \psi'_\mu = 0$

iii) The constraint gives us  $\partial_k \psi_m = \partial_m \chi_k$ . From (5.21) this implies

$\partial_k \psi_m = \epsilon_{km} b$ . Then the solution of this may be written as

$$\psi_m = \psi_m^{(0)} - \theta_m b \quad (5.25)$$

iv) The final constraint has the form

$$-\theta_k \partial_l \psi_m - \theta_l \partial_k \psi_m - \eta_{km} \psi_l - \eta_{lm} \psi_k = \partial_m \chi_{kl} - \Sigma_{kl} \psi_m$$

Substituting equations (5.25) and (5.22) into this we obtain

$$\Sigma_{kl} \psi_m^{(0)} - \eta_{km} \psi_l^{(0)} - \eta_{lm} \psi_k^{(0)} = -\Sigma_{kl} \phi_m + \eta_{km} \phi_l + \eta_{lm} \phi_k$$

This implies that

$$\psi_m = \psi_m - \phi_m - \theta_m b \quad (5.26)$$

where  $\Sigma_{kl} \psi_m - \eta_{km} \psi_l - \eta_{lm} \psi_k = 0$ .

This completes our section on the solutions to the constraint equations.

In the next section we will consider the choice of a six dimensional action and its reduction to four dimensions.

#### 5.4 BRST INVARIANT ACTION

In four dimensions the Rarita-Schwinger Lagrangian [15] is

$$L_{RS} = -i\bar{\Psi}_\lambda \gamma^{\lambda\mu\nu} \partial_\mu \Psi_\nu + \text{hermitian conjugate}$$

where

$$\gamma^{\lambda\mu\nu} = \gamma^\lambda \gamma^\mu \gamma^\nu - \gamma^\lambda g^{\mu\nu} + \gamma^\mu g^{\lambda\nu} - \gamma^\nu g^{\lambda\mu}$$

is antisymmetric under the interchange of any two indices.

A natural extension of this to six dimensions is

$$L_{RS} = i[L]\bar{\Psi}_L \Gamma^{LMN} \nabla_N \Psi_M (\text{sdet}E)/2 + \text{hermitian conjugate} \quad (5.27)$$

In this expression  $\Gamma^{LMN}$  is graded antisymmetric

$$\Gamma^{LMN} = \Gamma^L \Gamma^M \Gamma^N - \Gamma^L g^{MN} + [LM] \Gamma^M g^{LN} - [LN] [LM] \Gamma^N g^{LM}$$

provided the  $\Gamma^M$  satisfy (5.12) and we have  $\text{sdet}E = 1$  in our case.

(5.27) is invariant under gauge transformations  $\Psi_M \rightarrow \Psi_M + \nabla_M \chi$  provided the background geometry is Ricci flat which is true in our case.

Now  $\Psi$  is a spinor in an infinite-dimensional representation of  $OSp(4/2)$ . We can express  $\bar{\Psi}$  as a sum over a complete set of basis states

$$\bar{\Psi} = \sum_{ij} |\bar{i}\rangle \eta_{ij} \langle j| \bar{\Psi} \quad (5.28)$$

where  $\eta_{ij} = \langle i|j\rangle$  and  $\sum_{ij} |\bar{i}\rangle \eta_{ij} \langle j| = 1$ .

We then define  $\bar{\Psi}$  the conjugate spinor to  $\Psi$  by  $\bar{\Psi} = \Psi^\dagger \gamma_0$ . The requirement that  $\bar{\Psi}\Psi$  be invariant under  $OSp(4/2)$  transformations generated by  $\Sigma_{KL}$  (defined in (5.11)) imposes the condition

$$\Gamma_N^\dagger \gamma_0 = [N] \gamma_0 C_N^M \Gamma_M \quad (5.29)$$

where  $C_N^M$  is defined by  $(X^M)^\dagger = X^N C_N^M$  i.e.  $C_\mu^\nu = \delta_\mu^\nu$  and  $C_n^m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and we assume  $\Gamma_\mu^\dagger \gamma_0 = \gamma_0 \Gamma_\mu$ . For more details see Appendix E. In fact  $\bar{\Psi}\Psi$  is invariant under  $\Psi \rightarrow \Psi + X\Psi$  provided  $\gamma_0 X^\dagger \gamma_0 = -X$ .

In (5.27) we define  $\bar{\Psi}_L$  by  $\overline{\chi^L \Psi_L} = \bar{\Psi}_L \chi_L$  i.e.  $\bar{\Psi}^M = \bar{\Psi}^N C_N^M$

This definition of  $\bar{\Psi}_L$  is consistent with  $\overline{\Gamma^L \Psi_L} = \bar{\Psi}_L \Gamma_L$  as the conjugate spinor to  $\Gamma^L \Psi_L$ . Hence (5.27) is  $OSp(4/2)$  invariant.

To proceed further we need to choose a representation for the generalised Dirac matrices,  $\Gamma_M$ , and their action on the basis states.

We choose

$$\Gamma^M = (\Gamma^\mu, \Gamma^m) = (\gamma^\mu \times 1, i\sqrt{2}\gamma_5 \alpha^m) \quad (5.30)$$

where  $a^m = (a^5, a^6) = (a, a^\dagger)$  and  $[a_m, a_n] = \eta_{mn}$ . This choice is consistent with (5.12) and (5.29). Our ordinary Dirac matrices are given in Appendix A. Now we need only fix the action of the  $a^m$  on the basis states. We take the action of  $a^5$  on a state  $|i\rangle$  to lower  $i$  by one i.e.  $a|i\rangle = q|i-1\rangle$  and the action of  $a^\dagger$  to be  $a^\dagger|i\rangle = q'|i+1\rangle$  where  $q$  and  $q'$  are appropriate real normalisation factors. We also choose  $|\eta_{ij}| = \delta_{ij}$  for the metric  $\eta_{ij}$ .

Normally one would then choose a state  $|0\rangle$  to be annihilated by  $a$  i.e.  $a|0\rangle = 0$ ,  $a^\dagger a|0\rangle = 0$ . However we wish to have solutions of  $\Sigma_{mn} \bar{\Psi}_\mu = 0$  and  $\Sigma_{mn} \bar{\Psi}_k - \eta_{mk} \bar{\Psi}_n - \eta_{nk} \bar{\Psi}_m = 0$  where  $\Sigma_{mn} = -\{\Gamma_m, \Gamma_n\}/4$ .

We choose a state  $|c\rangle$  to satisfy  $\Sigma_{56}|c\rangle = -(a^\dagger a + \frac{1}{2})|c\rangle = 0$ ,  $a^\dagger a|c\rangle = c|c\rangle$  and  $\langle c|c\rangle = 1$ . This implies  $c = -\frac{1}{2}$  and an infinite dimensional representation. In Appendix F we derive the action of the generalised Dirac matrices on the basis states and the form of the metric. We state the results of this below.

We find the action of  $a^m$  on a basis state to be

$$a^\dagger | -n - \frac{1}{2} \rangle = (n + \frac{1}{2})^{\frac{1}{2}} | -n-1-\frac{1}{2} \rangle; \quad n \geq 0$$

$$a^\dagger | n - \frac{1}{2} \rangle = (n + \frac{1}{2})^{\frac{1}{2}} | n+1-\frac{1}{2} \rangle; \quad n \geq 0$$

$$a | n - \frac{1}{2} \rangle = (n - \frac{1}{2})^{\frac{1}{2}} | n-1-\frac{1}{2} \rangle; \quad n > 0$$

$$a^\dagger | -n - \frac{1}{2} \rangle = - (n - \frac{1}{2})^{\frac{1}{2}} | -n+1-\frac{1}{2} \rangle; \quad n > 0$$

The metric is given by

$$\eta_{1-\frac{1}{2} \ 1-\frac{1}{2}} = (-1)^{(|1|-1)/2} \quad (5.31)$$

Finally the matrix elements  $\langle i | a^m | j \rangle$  are

$$\begin{aligned} \langle n+\frac{1}{2} | a^\dagger | n-\frac{1}{2} \rangle &= (n+\frac{1}{2})^{\frac{1}{2}} \\ \langle -m+\frac{1}{2} | a^\dagger | -m-\frac{1}{2} \rangle &= (-1)^m (m-\frac{1}{2})^{\frac{1}{2}} \\ \langle n-\frac{1}{2} | a | n+\frac{1}{2} \rangle &= (n+\frac{1}{2})^{\frac{1}{2}} \\ \langle -m-\frac{1}{2} | a | -m+\frac{1}{2} \rangle &= (-1)^m (m-\frac{1}{2})^{\frac{1}{2}} \end{aligned} \quad (5.32)$$

where  $n \geq 0$  and  $m > 0$ .

Let us define  $\psi^n = \langle n-\frac{1}{2} | \psi \rangle$ . Then our choice of  $|c\rangle$  tells us  $\psi_\mu^0$  is a solution of  $\Sigma_{mn} \psi_\mu = 0$ . For convenience we then define  $\psi_\mu = \psi_\mu^0 + \partial_\mu \phi^0$ .

Now if  $m = n$ ,  $\Sigma_{mn} \psi_k - \eta_{mk} \psi_n - \eta_{nk} \psi_m = 0$  becomes  $\Sigma_{mm} \psi_k - 2\eta_{mk} \psi_m = 0$

which has no non-trivial solutions. If we take  $m \neq n$  then we require

$$\Sigma_{56} \psi_5 - \eta_{65} \psi_5 = -(a^\dagger a - \frac{1}{2}) \psi = 0$$

$$\Sigma_{56} \psi_6 - \eta_{56} \psi_6 = -(a^\dagger a + \frac{3}{2}) \psi_6 = 0$$

$\psi_5^{+1}$  and  $\psi_6^{-1}$  are solutions by (5.32).

Our action for the Rarita-Schwinger field is given by  $\int d^6 X X^2 L_{RS}$  and we now have sufficient information to find the action in four dimensions. We simply write  $L_{RS}$  in terms of the superfield components and integrate over the extra two dimensions. Before we do so it is helpful to use the gauge invariance of (5.27) as already noted. Now our solutions (5.24) and (5.26) can be written as

$$\psi_M = \psi_M + \partial_M (-\theta^q \psi_q + \frac{1}{2} \theta^q \theta_q b)$$

where  $\psi_M \neq 0$  only for  $\psi_\mu$ ,  $\psi_5^{+1}$  and  $\psi_6^{-1}$ . Thus we can choose a gauge in which our solutions involve only

$$\psi_\mu^0 = \psi_\mu, \psi_5^{+1} = \psi_5^{+1}, \psi_6^{-1} = \psi_6^{-1}$$

and have no  $\theta$  dependence.

Now in the expansion of  $L_{RS}$  (5.27) we find

$$\begin{aligned}
 [L]\bar{\Psi}_L \Gamma^{LMN} \nabla_N \Psi_M &= \bar{\Psi}_\lambda \gamma^{\lambda\mu\nu} \partial_\nu \Psi_\mu + \bar{\Psi}_\lambda (\gamma^\lambda \Gamma^m \gamma^\nu + \Gamma^m g^{\lambda\nu}) \partial_\nu \Psi_m - \bar{\Psi}_5 (\Gamma^5 \gamma^\mu \gamma^\nu - \Gamma^5 g^{\mu\nu}) \partial_\nu \Psi_\mu \\
 &- \bar{\Psi}_6 (\Gamma^6 \gamma^\mu \gamma^\nu - \Gamma^6 g^{\mu\nu}) \partial_\nu \Psi_\mu - \bar{\Psi}_5 (\Gamma^5 \Gamma^m - g^{5m}) \not\partial \Psi_m - \bar{\Psi}_6 (\Gamma^6 \Gamma^m - g^{6m}) \not\partial \Psi_m
 \end{aligned} \quad (5.33)$$

For convenience we note here that

$$g^{56} = -1 = -g^{65}, \quad \bar{\Psi}_5 = -\bar{\Psi}_6, \quad \bar{\Psi}_6 = -\bar{\Psi}_5, \quad \Gamma^5 = i\sqrt{2}\gamma_5 a, \quad \Gamma^6 = i\sqrt{2}\gamma_a a^\dagger$$

Then for example  $-\bar{\Psi}_5 (\Gamma^5 \Gamma^m - g^{5m}) \not\partial \Psi_m$  becomes

$$\begin{aligned}
 &-2\bar{\Psi}_6^{-1} \eta_{-3/2-3/2} < -3/2 | a | -\frac{1}{2} \eta_{-\frac{1}{2}-\frac{1}{2}} < -\frac{1}{2} | a | \frac{1}{2} \eta_{\frac{1}{2}\frac{1}{2}} \not\partial \psi_5^{+1} + \bar{\Psi}_6^{-1} \eta_{-3/2-3/2} \not\partial \psi_6^{-1} \\
 &-2\bar{\Psi}_6^{-1} \eta_{-3/2-3/2} < -3/2 | a | -\frac{1}{2} \eta_{-\frac{1}{2}-\frac{1}{2}} < -\frac{1}{2} | a^\dagger | -3/2 \eta_{-3/2-3/2} \not\partial \psi_6^{-1}
 \end{aligned}$$

which reduces to  $-\bar{\Psi}_6^{-1} \not\partial \psi_5^{+1} - 2\bar{\Psi}_6^{-1} \not\partial \psi_6^{-1}$  by (5.31) and (5.32).

In this way we find (5.33) becomes

$$\begin{aligned}
 &-\bar{\Psi}_\lambda \gamma^{\lambda\mu\nu} \partial_\mu \Psi_\nu - i(\bar{\Psi} \cdot \gamma \not\partial - \bar{\Psi} \cdot \partial) \gamma_5 (\psi_5^{+1} + \psi_6^{-1}) - i(\bar{\Psi}_5^{+1} + \bar{\Psi}_6^{-1}) \gamma_5 (\not\partial \gamma \cdot \psi - \partial \cdot \psi) \\
 &-3(\bar{\Psi}_5^{+1} + \bar{\Psi}_6^{-1}) \not\partial (\psi_5^{+1} + \psi_6^{-1}) / 2 - \frac{1}{2} (\bar{\Psi}_5^{+1} - \bar{\Psi}_6^{-1}) \not\partial (\psi_5^{+1} - \psi_6^{-1})
 \end{aligned}$$

Hence the action (5.27) after dimensional reduction is

$$\begin{aligned}
 \int d^6 X X^2 L_{RS} &= \int d^4 X [-i\bar{\Psi}_\lambda \gamma^{\lambda\mu\nu} \partial_\mu \Psi_\nu + (\bar{\Psi} \cdot \gamma \not\partial - \bar{\Psi} \cdot \partial) \gamma_5 (\psi_5^{+1} + \psi_6^{-1}) \\
 &+ (\bar{\Psi}_5^{+1} + \bar{\Psi}_6^{-1}) \gamma_5 (\not\partial \gamma \cdot \psi - \partial \cdot \psi) - 3i(\bar{\Psi}_5^{+1} + \bar{\Psi}_6^{-1}) \not\partial (\psi_5^{+1} + \psi_6^{-1}) / 2 \\
 &- i(\bar{\Psi}_5^{+1} - \bar{\Psi}_6^{-1}) \not\partial (\psi_5^{+1} - \psi_6^{-1}) / 2]
 \end{aligned} \quad (5.34)$$

Having calculated the gauge invariant part of the action we must now choose a gauge-fixing term. We seek a term of the form  $\bar{\Psi}^M T_M^N \Psi_N$  where  $T_M^N$  can be constructed from generalised Dirac matrices. We find two such terms and write

$$\begin{aligned}
 L_{GF} &= L_{GF1} + L_{GF2} \\
 \text{where} \quad \mu_1 L_{GF1} &= 2\bar{\Psi}^M \Psi_M
 \end{aligned} \quad (5.35)$$



$$\mu_2 L_{GF2} = \overline{\gamma \cdot \psi \gamma \cdot \psi} + \alpha \overline{\gamma \cdot \psi \Gamma^m \psi_m} + \beta \overline{\Gamma^m \psi_m \gamma \cdot \psi} + \rho \overline{\Gamma^m \cdot \psi \Gamma^n \psi_n} \quad (5.36)$$

where  $\mu_1$  and  $\mu_2$  are gauge-fixing parameters with dimensions of mass.

We now calculate the contributions of these terms to the action separately.

Expanding the first gauge-fixing term (5.35) we have

$$\int d^6 x \mu_1 L_{GF1} = \int d^6 x 2 (\overline{\psi}^\mu \psi_\mu + \overline{\psi}_5 \psi_5 - \overline{\psi}_6 \psi_6)$$

Calculating each term individually we have from (5.28), (5.24) and (5.26)

$$\begin{aligned} \overline{\psi}^\mu \psi_\mu &= \overline{\psi}^\mu \psi_\mu^0 + \sum_{n \neq 0} \overline{\psi}^\mu n \eta_{n-\frac{1}{2}n-\frac{1}{2}} \psi_\mu^n \\ &= (\overline{\psi}^\mu - \partial^\mu \overline{\phi}_5^0 \overline{\theta} - \partial^\mu \overline{\phi}_6^0 \overline{\theta} - \partial^\mu \overline{b}^0 \overline{\theta} \overline{\theta}) (\psi_\mu - \theta^q \partial_\mu \phi_q^0 - \theta \overline{\theta} \partial_\mu b^0) \\ &+ \sum_{n \neq 0} (\partial^\mu \overline{\phi}_5^n - \partial^\mu \overline{\phi}_5^0 \overline{\theta} - \partial^\mu \overline{\phi}_6^n \overline{\theta} - \partial^\mu \overline{b}^n \overline{\theta} \overline{\theta}) \eta_{n-\frac{1}{2}n-\frac{1}{2}} (\partial_\mu \phi^n - \theta^q \partial_\mu \phi_q^n - \theta \overline{\theta} \partial_\mu b^n) \end{aligned}$$

Then since only terms of order  $\theta \overline{\theta}$  can contribute we have

$$\begin{aligned} \int d^6 x 2 \overline{\psi}^\mu \psi_\mu &= \int d^4 x \overline{\psi}^0 \partial \cdot \psi + \partial \cdot \overline{\psi} b^0 + \overline{\phi}_5^0 \partial^2 \phi_5^0 - \overline{\phi}_6^0 \partial^2 \phi_6^0 \\ &+ \sum_{n \neq 0} \eta_{n-\frac{1}{2}n-\frac{1}{2}} (\overline{\phi}_5^n \partial^2 \phi_5^n - \overline{\phi}_6^n \partial^2 \phi_6^n + \overline{\phi}^n \partial^2 b^n + \overline{b}^n \partial^2 \phi^n) \end{aligned}$$

The next term gives

$$\begin{aligned} \int d^6 x 2 \overline{\psi}_5 \psi_5 &= 2 \int d^6 x \overline{\psi}_5^{+1} \eta_{\frac{1}{2}2} \psi_5^{+1} + \sum_{n \neq 1} \overline{\psi}_5^n \eta_{n-\frac{1}{2}n-\frac{1}{2}} \psi_5^n \\ &= 2 \int d^6 x (\overline{\psi}_5^{+1} - \overline{\phi}_5^{+1} - \overline{b}^{+1} \overline{\theta}) (\psi_5^{+1} - \phi_5^{+1} - \overline{\theta} b^{+1}) + \sum_{n \neq 1} \eta_{n-\frac{1}{2}n-\frac{1}{2}} (\overline{\phi}_5^n - \overline{b}^n \overline{\theta}) (-\phi_5^n - \overline{\theta} b^n) \\ &= \int d^4 x \overline{b}^{+1} b^{+1} + \sum_{n \neq 1} \eta_{n-\frac{1}{2}n-\frac{1}{2}} \overline{b}^n b^n \end{aligned}$$

The final term is

$$\int d^6 x -2 \overline{\psi}_6 \psi_6 = \int d^4 x -\overline{b}^{-1} b^{-1} + \sum_{n \neq -1} \eta_{n-\frac{1}{2}n-\frac{1}{2}} \overline{b}^n b^n$$

Adding these terms together we obtain for the first gauge fixing term

(5.35)

$$\int d^6 x \mu_1 L_{GF1} = \int d^4 x b^0 \partial_\alpha \psi + \partial_\alpha \bar{\psi} b^0 + \phi_5^0 \partial^2 \phi_5^0 - \phi_6^0 \partial^2 \phi_6^0 + 2b^0 b^0 \quad (5.37)$$

$$+ \sum_{n \neq 0} n_{n-\frac{1}{2}n-\frac{1}{2}} (\bar{\phi}_5^n \partial^2 \phi_5^n - \bar{\phi}_6^n \partial^2 \phi_6^n + \bar{\phi}^n \partial^2 b^n + \bar{b}^n \partial^2 \phi^n + 2\bar{b}^n b^n)$$

Expanding the second gauge fixing term (5.36) we have

$$\frac{1}{2} \mu_2 L_{GF2} = \overline{\gamma \cdot \psi} \gamma \cdot \psi + \alpha \overline{\gamma \cdot \psi} \Gamma^m \psi_m + \beta \overline{\Gamma^m \psi_m} \gamma \cdot \psi + \rho \overline{\Gamma^m \psi_m} \Gamma^n \psi_n$$

$$= F^0 + F^+ + F^-$$

where

$$F^0 = (\overline{\gamma \cdot \psi})^0 (\gamma \cdot \psi)^0 + \alpha (\overline{\gamma \cdot \psi})^0 (\Gamma^m \psi_m)^0 + \beta (\overline{\Gamma^m \psi_m})^0 (\gamma \cdot \psi)^0 + \rho (\overline{\Gamma^m \psi_m})^0 (\Gamma^n \psi_n)^0$$

$$F^+ = \sum_{n>0} (\overline{\gamma \cdot \psi})^n (\gamma \cdot \psi)^n + \alpha (\overline{\gamma \cdot \psi})^n (\Gamma^m \psi_m)^n + \beta (\overline{\Gamma^m \psi_m})^n (\gamma \cdot \psi)^n + \rho (\overline{\Gamma^m \psi_m})^n (\Gamma^p \psi_p)^n$$

$$F^- = \sum_{m>0} n_{-m-\frac{1}{2}-m-\frac{1}{2}} [(\overline{\gamma \cdot \psi})^{-m} (\gamma \cdot \psi)^{-m} + \alpha (\overline{\gamma \cdot \psi})^{-m} (\Gamma^n \psi_n)^{-m}$$

$$+ \beta (\overline{\Gamma^n \psi_n})^{-m} (\gamma \cdot \psi)^{-m} + \rho (\overline{\Gamma^n \psi_n})^{-m} (\Gamma^p \psi_p)^{-m}]$$

Clearly we need to find  $(\gamma \cdot \psi)^i$  and  $(\Gamma^1 \psi_1)^i$  for  $i = 0, n, -m$ .

We give these below using the solutions (5.24) and (5.26).

$$(\gamma \cdot \psi)^0 = \gamma \cdot \psi - \theta \not{\partial} \phi_5^0 - \bar{\theta} \not{\partial} \phi_6^0 - \theta \bar{\theta} \not{\partial} b^0$$

$$(\gamma \cdot \psi)^n = \not{\partial} \phi^n - \theta \not{\partial} \phi_5^n - \bar{\theta} \not{\partial} \phi_6^n - \theta \bar{\theta} \not{\partial} b^n$$

$$(\gamma \cdot \psi)^{-m} = \not{\partial} \phi^{-m} - \theta \not{\partial} \phi_5^{-m} - \bar{\theta} \not{\partial} \phi_6^{-m} - \theta \bar{\theta} \not{\partial} b^{-m}$$

$$(\Gamma^1 \psi_1)^0 = (i\sqrt{2}a^\dagger \gamma_5 \psi_6 + i\sqrt{2}a \gamma_5 \psi_5)^0$$

$$= i\sqrt{2} \gamma_5 \langle -\frac{1}{2} | a^\dagger | j \rangle \eta_{jk} \psi_6^{k+\frac{1}{2}} + i\sqrt{2} \gamma_5 \langle -\frac{1}{2} | a | j \rangle \eta_{jk} \psi_5^{k+\frac{1}{2}}$$

$$= i\gamma_5 \psi_6^{-1} + i\gamma_5 \psi_5^{+1} \quad \text{by (5.31) and (5.32)}$$

$$= i\gamma_5 (\psi_6^{-1} - \phi_6^{-1} + \theta b^{-1} + \psi_5^{+1} - \phi_5^{-1} - \bar{\theta} b^{+1})$$

Similarly

$$\begin{aligned}
 (\Gamma^1 \Psi_1)^n &= i\sqrt{2}\gamma_5 \langle n-\frac{1}{2} | a^\dagger | j \rangle \eta_{jk} \Psi_5^{k+\frac{1}{2}} + i\sqrt{2}\gamma_5 \langle n-\frac{1}{2} | a | j \rangle \eta_{jk} \Psi_5^{k+\frac{1}{2}} \\
 &= i\gamma_5 (2n-1)^{\frac{1}{2}} \Psi_6^{n-1} + i\gamma_5 (2n+1)^{\frac{1}{2}} \Psi_5^{n+1} \\
 &= i\gamma_5 (2n-1)^{\frac{1}{2}} (-\phi_6^{n-1} + \theta b^{n-1}) + i\gamma_5 (2n+1)^{\frac{1}{2}} (-\phi_5^{n+1} - \bar{\theta} b^{n+1})
 \end{aligned}$$

and

$$(\Gamma^1 \Psi_1)^{-m} = i\gamma_5 (2m+1)^{\frac{1}{2}} (-\phi_5^{-m-1} + \theta b^{-m-1}) - i\gamma_5 (2m-1)^{\frac{1}{2}} (-\phi_5^{-m+1} - \bar{\theta} b^{-m+1})$$

These expressions then give us

$$\begin{aligned}
 F^0 &= (\overline{\gamma \cdot \Psi})^0 (\gamma \cdot \Psi)^0 + \alpha (\overline{\gamma \cdot \Psi})^0 (\Gamma^m \Psi_m)^0 + \beta (\overline{\Gamma^m \Psi_m})^0 (\gamma \cdot \Psi)^0 + \rho (\overline{\Gamma^m \Psi_m})^0 (\Gamma^n \Psi_n)^0 \\
 &= (\overline{\Psi} \cdot \gamma - \partial_\mu \overline{\phi}_5^0 \gamma^\mu \bar{\theta} - \partial_\mu \overline{\phi}_6^0 \gamma^\mu \theta - \partial_\mu \overline{b}^0 \gamma^\mu \bar{\theta} \bar{\theta}) (\gamma \cdot \psi - \theta \not{\phi}_5^0 - \bar{\theta} \not{\phi}_6^0 - \bar{\theta} \bar{\theta} \not{b}^0) + \\
 &+ \alpha (\overline{\Psi} \cdot \gamma - \partial_\mu \overline{\phi}_5^0 \gamma^\mu \bar{\theta} - \partial_\mu \overline{\phi}_6^0 \gamma^\mu \theta - \partial_\mu \overline{b}^0 \gamma^\mu \bar{\theta} \bar{\theta}) i\gamma_5 (\psi_5^{+1} - \phi_5^{+1} - \theta b^{+1} + \psi_6^{-1} - \phi_6^{-1} + \theta b^{-1}) \\
 &+ \beta (\overline{\psi}_5^{+1} - \overline{\phi}_5^{+1} - \overline{b}^{+1} \theta - \overline{\psi}_6^{-1} + \overline{\phi}_6^{-1} - \overline{b}^{-1} \bar{\theta}) i\gamma_5 (\gamma \cdot \psi - \theta \not{\phi}_5^0 - \bar{\theta} \not{\phi}_6^0 - \bar{\theta} \bar{\theta} \not{b}^0) \\
 &- \rho (\overline{\psi}_5^{+1} - \overline{\phi}_5^{+1} - \overline{b}^{+1} \theta - \overline{\psi}_6^{-1} + \overline{\phi}_6^{-1} - \overline{b}^{-1} \bar{\theta}) (\psi_5^{+1} - \phi_5^{+1} - \theta b^{+1} - \psi_6^{-1} + \phi_6^{-1} - \theta b^{-1})
 \end{aligned}$$

hence

$$\begin{aligned}
 2 \int d^6 X F^0 &= \int d^4 x \left[ \overline{b}^0 \not{\gamma} \cdot \psi - \overline{\psi} \cdot \gamma \not{b}^0 + \overline{\phi}_5^0 \partial^2 \phi_5^0 - \overline{\phi}_6^0 \partial^2 \phi_6^0 \right. \\
 &+ \alpha [i \overline{b}^0 \not{\gamma}_5 (\psi_5^{+1} + \psi_6^{-1}) - i \overline{b}^0 \not{\gamma}_5 \phi_5^{+1} - i \overline{b}^0 \not{\gamma}_5 \phi_6^{-1} - i \overline{\phi}_5^0 \not{\gamma}_5 b^{-1} - i \overline{\phi}_6^0 \not{\gamma}_5 b^{+1} \\
 &+ \beta [-i (\overline{\psi}_5^{+1} + \overline{\psi}_6^{-1}) \gamma_5 \not{b}^0 + i \overline{\phi}_5^{+1} \gamma_5 \not{b}^0 + i \overline{\phi}_6^{-1} \gamma_5 \not{b}^0 + i \overline{b}^{-1} \gamma_5 \not{\phi}_5^0 + i \overline{b}^{+1} \gamma_5 \not{\phi}_6^0] \\
 &\left. + \rho [-\overline{b}^1 b^1 + \overline{b}^{-1} b^{-1}] \right] \quad (5.38)
 \end{aligned}$$

Similarly

$$\begin{aligned}
 2 \int d^6 X F^+ &= \int d^4 x \sum_{n>0} \left[ \overline{b}^n \partial^2 \phi^n + \overline{\phi}^n \partial^2 b^n + \overline{\phi}_5^n \partial^2 \phi_5^n - \overline{\phi}_6^n \partial^2 \phi_6^n \right. \\
 &+ \alpha [-i (2n-1)^{\frac{1}{2}} (\overline{b}^n \not{\gamma}_5 \phi_5^{n-1} + \overline{\phi}_5^n \not{\gamma}_5 b^{n-1}) - i (2n+1)^{\frac{1}{2}} (\overline{b}^n \not{\gamma}_5 \phi_5^{n+1} + \overline{\phi}_6^n \not{\gamma}_5 b^{n+1})] \\
 &+ \beta [i (2n-1)^{\frac{1}{2}} (\overline{\phi}_5^{n-1} \gamma_5 \not{b}^n + \overline{b}^{n-1} \gamma_5 \not{\phi}_5^n) + i (2n+1)^{\frac{1}{2}} (\overline{\phi}_5^{n+1} \gamma_5 \not{b}^n + \overline{b}^{n+1} \gamma_5 \not{\phi}_6^n)] \\
 &\left. + \rho [-(2n+1) \overline{b}^{n+1} b^{n+1} + (2n-1) \overline{b}^{n-1} b^{n-1}] \right]
 \end{aligned}$$

which we write

$$\begin{aligned}
 & \int d^4x \sum_{n \geq 0} [ \bar{b} \partial^2 \phi^n + \bar{\phi} \partial^2 b^n + \bar{\phi}_5 \partial^2 \phi_5^n - \bar{\phi}_6 \partial^2 \phi_6^n \\
 & - i(2n+1)^{\frac{1}{2}} ( \alpha \bar{b} \not{\gamma}_5 \phi_5^{n+1} + \alpha \bar{\phi}_6 \not{\gamma}_5 b^{n+1} - \beta \bar{\phi}_5 \not{\gamma}_5 b^n - \beta \bar{b} \not{\gamma}_5 \phi_6^n ) \\
 & - i(2n+1)^{\frac{1}{2}} ( \alpha \bar{b} \not{\gamma}_5 \phi_6^{n+1} + \alpha \bar{\phi}_5 \not{\gamma}_5 b^{n+1} - \beta \bar{\phi}_6 \not{\gamma}_5 b^n - \beta \bar{b} \not{\gamma}_5 \phi_5^{n+1} ) \\
 & - \rho(2n+1) \bar{b}^{n+1} b^{n+1} + \rho(2n+1) \bar{b}^n b^n ] \\
 & - i( \alpha \bar{b} \not{\gamma}_5 \phi_6^0 + \alpha \bar{\phi}_5 \not{\gamma}_5 b^0 - \beta \bar{\phi}_6 \not{\gamma}_5 b^1 - \beta \bar{b} \not{\gamma}_5 \phi_5^1 ) + \rho \bar{b}^0 b^0
 \end{aligned}$$

In the same way we expand and rewrite  $F^-$ . The result is

$$\begin{aligned}
 2 \int d^6x F^- &= \int d^4x \sum_{m \geq 0} (-1)^m [ \bar{b} \partial^2 \phi^{-m} + \bar{\phi} \partial^2 b^{-m} + \bar{\phi}_5 \partial^2 \phi_5^{-m} - \bar{\phi}_6 \partial^2 \phi_6^{-m} \\
 & - i(2m+1)^{\frac{1}{2}} ( \alpha \bar{b} \not{\gamma}_5 \phi_6^{-m-1} + \alpha \bar{\phi}_5 \not{\gamma}_5 b^{-m-1} - \beta \bar{\phi}_6 \not{\gamma}_5 b^{-m} - \beta \bar{b} \not{\gamma}_5 \phi_5^{-m} ) \\
 & - i(2m+1)^{\frac{1}{2}} ( \alpha \bar{b} \not{\gamma}_5 \phi_5^{-m-1} + \alpha \bar{\phi}_6 \not{\gamma}_5 b^{-m-1} - \beta \bar{\phi}_5 \not{\gamma}_5 b^{-m} - \beta \bar{b} \not{\gamma}_5 \phi_6^{-m-1} ) \\
 & + \rho(2m+1) \bar{b}^{-m-1} b^{-m-1} + \rho(2m+1) \bar{b}^{-m} b^{-m} ] \\
 & - i( \alpha \bar{b} \not{\gamma}_5 \phi_5^0 + \alpha \bar{\phi}_6 \not{\gamma}_5 b^0 - \beta \bar{\phi}_5 \not{\gamma}_5 b^{-1} - \beta \bar{b} \not{\gamma}_5 \phi_6^{-1} ) + \rho \bar{b}^0 b^0
 \end{aligned}$$

Now in these three parts of the second gauge-fixing term there are terms which combine fields of non-zero ghost number with zero ghost number fields, for example  $\beta \bar{b} \not{\gamma}_5 \phi_6^0$  in (5.38). It seems if we take  $\alpha = -\beta$  we can cancel all these terms. However if (5.36) is to be non-hermitian then we require  $\alpha^* = \beta$ . Hence we take  $\alpha = -\beta = i\omega$ . The terms of mixed ghost number vanish and we are left with

$$\begin{aligned}
 \int d^6x \mu_2 L_{GF2} &= \int d^4x \bar{b} \not{\gamma}_5 \psi \bar{\psi} \not{\gamma}_5 b^0 + \bar{\phi}_5 \partial^2 \phi_5^0 - \bar{\phi}_6 \partial^2 \phi_6^0 + 2\rho \bar{b}^0 b^0 \\
 & - \omega \bar{b} \not{\gamma}_5 (\psi_5^+ + \psi_6^-) - \omega (\bar{\psi}_5^+ + \bar{\psi}_6^-) \not{\gamma}_5 b^0 \\
 & + \sum_{n \neq 0} n_{n-\frac{1}{2}, n-\frac{1}{2}} ( \bar{\phi}_5 \partial^2 \phi_5^n - \bar{\phi}_6 \partial^2 \phi_6^n + \bar{\phi} \partial^2 b^n + \bar{b} \partial^2 \phi^n + 2\rho \bar{b}^n b^n )
 \end{aligned} \tag{5.39}$$

Note, if instead of the gauge-fixing terms (5.35) and (5.36) we took ones of the form  $i\bar{\Psi}^M \Gamma_N^N \Psi_M$  and  $i\bar{\Gamma}_M^M \Psi_N^N \Psi_N$  [16]

then we would have obtained higher derivative ghost propagators.

## 5.5 DISCUSSION

It is a simple matter to verify that the total action is invariant under extended BRST transformations. The latter are given simply by the change in the fields under supertranslations. From (5.24) the variation of  $\Psi_\mu$  under  $\theta^m \rightarrow \theta^m + \epsilon^m$  is

$$\delta\Psi_\mu = -\epsilon^n \partial_\mu \phi_n - \theta \bar{\epsilon} \partial_\mu b - \bar{\epsilon} \theta \partial_\mu b$$

If we define the variation of the components of  $\Psi_\mu$  by

$$\delta\Psi_\mu = \delta\psi'_\mu + \partial_\mu \delta\phi - \theta^n \partial_\mu \delta\phi_n - \frac{1}{2} \theta^n \theta_n \partial_\mu \delta b$$

then we obtain

$$\delta b^n = 0$$

$$\delta\phi_5^n = \bar{\epsilon} b^n$$

$$\delta\phi_6^n = -\epsilon b^n$$

for all modes  $n$  and

$$\delta\phi^m = -\epsilon\phi_5^m - \bar{\epsilon}\phi_6^m$$

for  $m \neq 0$  where each variation is up to a constant which we have set equal to zero. Also

$$\delta\psi_\mu = -\epsilon \partial_\mu \phi_5^0 - \bar{\epsilon} \partial_\mu \phi_6^0$$

The variation of  $\Psi_m$  from (5.26) is

$$\delta\Psi_m = -\epsilon_m b$$

Defining the variation of the components of  $\bar{\Psi}_m$  as before we obtain

$$\delta\psi_m = 0$$

Our total action from (5.34), (5.37) and (5.39) is

$$S = \int d^4x -i\bar{\psi}_\lambda \gamma^{\lambda\mu\rho} \partial_\mu \psi_\rho + (\psi \cdot \gamma \not{\partial} \psi - \partial \cdot \psi) G/2 - G(\not{\partial} \gamma \cdot \psi - \partial \cdot \psi)/2 + \bar{c} \partial^2 i c_\star - i \bar{c}_\star \partial^2 c \quad (5.40)$$

$$+ [\bar{b}^0 \partial \cdot \psi - \bar{\psi} \cdot \partial b^0 - 2\bar{b}^0 b^0]/\mu_1 + [\bar{b}^0 \not{\partial} \gamma \cdot \psi - \bar{\psi} \cdot \gamma \not{\partial} b^0 - \omega \bar{b}^0 G/2 + \omega G \not{\partial} b^0/2 + 2\rho \bar{b}^0 b^0]/\mu_2$$

where  $G = 2\gamma_5(\psi_5^{+1} + \psi_6^{-1})$ ,  $C = (\phi_5^0 + \phi_6^0)(\mu_1 + \mu_2)/\mu_2\mu_1$ ,  $i c_\star = (\phi_5^0 - \phi_6^0)/2$  and

we have ignored terms which decouple from this. This action is not completely gauge-fixed. There is an invariance under  $\delta G = 4i\partial\Lambda$  and  $\delta\psi_\lambda = \gamma_\lambda \partial\Lambda + \sigma\partial_\lambda\Lambda$  for  $\sigma = \sigma(\omega, \mu_1, \mu_2)$  and some parameter  $\Lambda$ .

If we take  $\rho = -\mu_1/\mu_2$  then the  $\bar{b}^0$  equation of motion gives

$$\partial \cdot \psi = (-\partial \gamma \cdot \psi + \omega \not{\partial} G/2)\mu_1/\mu_2 \quad (5.41)$$

and similarly for  $\bar{b}^0$ . We can then write (5.40) as

$$S = \int d^4x -i\bar{\psi}_\lambda \gamma^{\lambda\mu\rho} \partial_\mu \psi_\rho + \bar{\psi} \gamma \not{\partial} G - \bar{G} \not{\partial} \gamma \cdot \psi - 3i\xi \bar{G} \not{\partial} G/8 + \bar{c} \partial^2 i c_\star - i \bar{c}_\star \partial^2 c \quad (5.42)$$

where  $\xi = 4\mu_2^2/(\mu_1 + \mu_2)^2$ . (5.42) is a particular case of the spinor sector of supergravity in references [8,9] with covariant gauge-fixing parameter  $3i\xi/8$ .  $c$  and  $c_\star$  are the Fadeev-Popov ghosts while a linear combination of  $G$  and  $\gamma \cdot \psi$  plays the role of the Nielsen-Kallosh ghost [7]. In reference [8], Hata and Kugo prove that the physical S-matrix is unitary for arbitrary covariant gauge-fixing. The inclusion of the auxiliary field  $G$  and assumption of Fadeev-Popov ghosts with propagator proportional to  $p^2$  was crucial to this proof. In our case these arise naturally as a consequence of our dimensional reduction procedure. However our action (5.42) is supplemented by the condition (5.41) i.e. it is invariant under the same transformation as (5.40).

The solution to this problem may lie in the study of more exotic gauge-fixing terms. This is an unappealing scenario for a method whose appeal, to date, lies in its simplicity.

Instead let us write  $L_{RS}$  and  $L_{GF2}$  as

$$L_{RS} = -i(\overline{\Psi_\lambda + \frac{1}{2}\gamma_\lambda \Gamma^m \Psi_m}) \gamma^{\lambda\mu\rho} \partial_\mu (\Psi_\rho + \frac{1}{2}\gamma_\rho \Gamma^n \Psi_n), \quad L_{GF2} = (\overline{\gamma \cdot \Psi + i \Gamma^m \Psi_m})(\gamma \cdot \Psi + i \Gamma^n \Psi_n) / \mu_2 \quad (5.43)$$

from (5.36) with  $\alpha = -\beta = i$  and  $\rho = 1$ . This form of  $L_{RS}$  was suggested by the authors of [15]. The presence of  $i$  in  $L_{GF2}$  is due to our requirement that it be hermitian and that terms with mixed ghost number should cancel. Then it is possible that the solution to our problem may lie, at a more fundamental level, in our choice of conjugation properties for  $\theta^m$  [17]. Using a different choice we might hope to write (5.43) as  $L_{GF2} = \overline{\Gamma^M \Psi'_M} \Gamma^N \Psi'_N / 2$  where  $\Psi'_M = (\Psi'_\mu, (\tau-1)\Psi_m)$  and  $\Psi'_\mu = \Psi_\mu + \frac{1}{2}\gamma_\mu \Gamma^m \Psi_m$ . After dimensional reduction our action  $S = \int d^6 X X^2 L_{RS} + L_{GF2}$  would become  $S = \int d^4 x -i \overline{\Psi'_\lambda} \gamma^{\lambda\mu\rho} \partial_\mu \Psi'_\rho + (\overline{b^0} \not{D} \gamma \cdot \Psi' - \overline{\Psi'} \cdot \gamma \not{D} b^0 + 2\tau \overline{b^0} b^0) / \mu_2$

This action has no residual invariance and is totally gauge fixed. It differs from (5.42) since  $b^0$  has different dimensions from  $G$ .

It should be emphasised that the final solutions of the constraint equations for  $\Psi_M$ , amount to a CSDR formalism where  $S/R$  is no longer  $Sp(2) \wedge T_2 / Sp(2)$  but simply  $U(1) \wedge T_2 / U(1)$ . The  $U(1)$  is generated by  $\Sigma_{56}$ . The spinor representations of  $OSp(4/2)$  which we use contain no  $Sp(2)$  singlets. In order to recover the four-dimensional Rarita-Schwinger field, it is crucial to seek singlets of just  $U(1)$ . In turn this requires modified Fock space representations. Note that the cases studied previously in BRST quantisation - Yang-Mills theory and in this thesis, antisymmetric tensor gauge fields - would go through equally well with this restricted coset-space.

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## 6 THE GRASSMANN EUCLIDEAN GROUP AND ITS REPRESENTATIONS

### 6.1 BRST SUPERALGEBRA

The BRST transformation [1] arises in the study of covariant, canonical quantisation of gauge theories. One starts with a gauge invariant Lagrangian. The requirement that this be in covariant form necessitates the inclusion of non-physical modes. In order to have well defined propagators in the theory one adds a gauge fixing term to the Lagrangian. However this gives the non-physical modes well defined propagators also. It is feared that these non-physical modes may propagate as virtual, intermediate states and violate unitarity because they often have negative norms. In order to reconcile unitary and covariance of the Lagrangian one adds a ghost term. The ghosts remove the degrees of freedom associated with the introduction of the non-physical modes. The total Lagrangian is then no longer invariant under a gauge transformation. Instead it is invariant under the global BRST transformation which mixes the gauge and ghost fields.

We denote the generator of this transformation by  $Q_B$  and take  $Q_B$  to be hermitian. The theory is also invariant under a scaling of the ghosts [2]. We denote the hermitian generator of this transformation by  $Q_C$ . Then the BRST algebra is given by

$$i[Q_C, Q_B] = Q_B, \quad Q_B^2 = 0 \quad (6.1)$$

It was then found that the roles of the ghost and anti-ghost fields could be interchanged [3]. The dual BRST transformation then mixes gauge and anti-ghost fields. Denoting the hermitian generator of this transformation by  $Q_{\bar{B}}$ , we obtain the extended BRST algebra by adding

$$i[Q_C, \bar{Q}_B] = -\bar{Q}_B, \bar{Q}_B^2 = \{Q_B, \bar{Q}_B\} = 0 \quad (6.2)$$

to (6.1).

The next step was made by Nakanishi and Ojima [4] who found that in the Landau gauge for Yang-Mills theory and quantum gravity a much larger symmetry group is admitted.

Nishijima in [5] then considered an enlarged algebra in which he included two of the generators found in [4]. These are  $Q = Q(c, c)$  and  $\bar{Q} = \bar{Q}(\bar{c}, \bar{c})$ . The additional commutation rules are

$$\begin{aligned} i[Q_C, Q] &= 2Q, \quad i[Q_C, \bar{Q}] = -2\bar{Q}, \quad [Q, \bar{Q}] = 4iQ_C \\ [Q, \bar{Q}_B] &= 2iQ_B, \quad [\bar{Q}, Q_B] = -2i\bar{Q}_B \end{aligned} \quad (6.3)$$

where  $Q$  and  $\bar{Q}$  are only conserved in the Landau gauge.

Finally, Delbourgo and Jarvis [6] introduced the group  $Sp(2) \wedge T_2$  whose generators obey

$$\begin{aligned} [J_{kl}, J_{mn}] &= i(\eta_{km}J_{ln} + \eta_{kn}J_{lm} + \eta_{lm}J_{kn} + \eta_{ln}J_{km}) \\ [J_{kl}, P_m] &= i(\eta_{km}P_l + \eta_{lm}P_k) \\ \{P_m, P_n\} &= 0 \end{aligned} \quad (6.4)$$

with  $k = 1, 2$ . The  $P_k$  generate supertranslations while the  $J_{Mn}$ , symmetric in their indices, rotate and scale the ghosts. Note that in this form our generators  $J$  differ by a factor of  $(-i)$  from those in previous chapters. However this chapter is completely self-contained and (6.4) provides a closer analogy with the familiar Poincaré group. The reality and finite dimensionality assignments

$$P_1^\dagger = P_2, \quad J_{11}^\dagger = J_{22}, \quad J_{12}^\dagger = -J_{12} \quad (6.5)$$

## 6.2 BASIS STATES

Since the BRST superalgebra is a fermionic analogue of the inhomogeneous rotation group we anticipate that its representations will be similar to those of the Poincaré group.

The Poincaré group representations are classified according to the values of two Casimir operators,  $P^2$  and  $W^2$ .  $P_\mu$  is the energy-momentum operator which generates translations.  $W_\mu$  is the Pauli-Lubanski vector. It is constructed from  $J_{\mu\nu}$ , the angular momentum operator and generator of infinitesimal Lorentz transformations by

$$W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$$

$W^2$  is related to the spin  $S$  by

$$W^2 = -M^2 S(S+1) \quad (6.6)$$

where  $M^2$  denotes the eigenvalue of  $P^2$ .

To label the representations of the BRST super-algebra we may use the (nilpotent) Casimir

$$P^2 = P^k P_k = 2P_2 P_1$$

and we try an analogue of the Pauli-Lubanski vector  $W^2 = \alpha W^{klm} W_{mkl}$

where  $\alpha$  is a constant and

$$\begin{aligned} W_{klm} &= P_k J_{lm} + P_l J_{mk} + P_m J_{kl} \\ &= J_{kl} P_m + J_{lm} P_k + J_{mk} P_l \quad \text{by (6.4).} \end{aligned}$$

Since the Casimirs which label an irreducible representation must be scalar and super-translation invariant we check that  $W_{klm}$  is translation invariant

$$\{W_{klm}, P_n\} = \{P_k J_{lm} + P_l J_{mk} + P_m J_{kl}, P_n\}$$

$$= P_k (i\eta_{ln} P_m + i\eta_{mn} P_l) + \{P_n, P_k\} J_{lm} + (k \rightarrow l, l \rightarrow m, m \rightarrow k) + (k \rightarrow m, l \rightarrow k, m \rightarrow l) \text{ by (6.4)}$$

$$= i\eta_{ln} P_k P_m + i\eta_{mn} P_k P_l + i\eta_{mn} P_l P_k + i\eta_{kn} P_l P_m + i\eta_{kn} P_m P_l + i\eta_{ln} P_m P_k \text{ by (6.4)}$$

Hence

$$\{W_{klm}, P_n\} = 0 \quad (6.7)$$

We note also that

$$\{W_{klm}, W_{pqr}\} = (i(P_p \eta_{qk} + P_q \eta_{pk}) W_{rlm} + (p, q, r)) + (k, l, m) \quad (6.8)$$

where for brevity we have introduced the notation

$$Z_{pqr} + (p, q, r) = Z_{pqr} + Z_{qrp} + Z_{rpq}$$

To prove (6.8) we have

$$\begin{aligned} \{W_{klm}, W_{pqr}\} &= \{W_{klm}, P_p J_{qr} + P_q J_{rp} + P_r J_{pq}\} \\ &= \{W_{klm}, P_p\} J_{qr} + P_p [J_{qr}, W_{klm}] + (p, q, r) \\ &= (P_p [J_{qr}, P_k J_{lm}] + (k, l, m)) + (p, q, r) \text{ by (6.7)} \\ &= (P_p (i\eta_{qk} P_r + i\eta_{rk} P_q) J_{lm} \\ &\quad + P_p P_k i(\eta_{ql} J_{rm} + \eta_{qm} J_{lr} + \eta_{rl} J_{qm} + \eta_{rm} J_{lq})) + (k, l, m) + (p, q, r) \end{aligned}$$

which can be rearranged to give the required result.

We also find

$$\begin{aligned} W^{klm} W_{mlk} &= (J^{kl} P^m + J^{lm} P^k + J^{mk} P^l) (P_m J_{lk} + P_l J_{km} + P_k J_{ml}) \\ &= 6P^m P_m J^{kl} J_{lk} \end{aligned} \quad (6.9)$$

$$\text{since } P^m P_k = \delta^m_k P^l P_l.$$

The spin  $s$  is defined as in (6.6) by the ratio of  $w^2$  to  $p^2$  that is

$$w^2 = p^2 s(s+1).$$

The case  $p^2 = 0$  is a special one and will be discussed in Section 6.4.

In Section 6.3 we will see that this implies  $w^2 = W^{klm} W_{mlk} / 48$ .

Now that we have found two labels,  $p^2$  and  $s$ , we need to consider how to label the rest of the basis state. There are at least two possibilities:

- i) The first is an analogue of the Poincaré states  $|s;p\rangle$  in which the direction of the momentum is included as a label. In our case we write  $|s;p\rangle$  where  $| \rangle$  indicates the representation is non-unitary. Since this is an eigenstate of  $p^2$  we have

$$P_k |s;p\rangle = p_k |s;p\rangle$$

From the commutation relations we have

$$\exp(i\alpha^{mn}J_{mn}/2)P_q\exp(-i\alpha^{kl}J_{kl}/2) = (\exp-\alpha)_q^k P_k$$

which implies

$$P_l \exp(-i\alpha^{mn}J_{mn}/2) |s;p\rangle = \exp(-i\alpha^{mn}J_{mn}) (\exp-\alpha)_l^k p_k |s;p\rangle$$

Hence the action of  $J_{mn}$  on the state is

$$\exp(-i\alpha^{mn}J_{mn}/2) |s;p\rangle = |s;\exp(-\alpha)p\rangle$$

Finally the action of a general  $Sp(2)$  transformation on the state is given by

$$\exp(-i\xi^l P_l) \exp(-i\alpha^{mn}J_{mn}/2) |s;p\rangle = \exp(-i\xi^l P_l) |s;\exp(-\alpha)p\rangle$$

- ii) Our second possibility is to replace 'the direction of  $p$ ' by a helicity variable  $\lambda$ . We take  $\lambda$  to be related to the eigenvalue of  $J_{12}$ . Then for our states  $|s;p^2;\lambda\rangle$  we have

$$J_{12} |s;p^2;\lambda\rangle = -2i\lambda |s;p^2;\lambda\rangle \quad (6.10)$$

At this point we note the isomorphism between  $Sp(2)$  and  $SU(2)$  via the identification of the generators

$$J_{12} \leftrightarrow -2iJ_3, J_{11} \leftrightarrow 2(J_1 + iJ_2), J_{22} \leftrightarrow 2(J_1 - iJ_2)$$

are adopted. Correspondence with previous algebras in this section is given by

$$Q_C \leftrightarrow J_{12}, Q_B \leftrightarrow P_1, \bar{Q}_B \leftrightarrow P_2, Q \leftrightarrow J_{11}, \bar{Q} \leftrightarrow J_{22}$$

Note that for this algebra  $Q$  and  $\bar{Q}$  are conserved for any  $Sp(2)$  invariant gauge-fixing term.

Having introduced our BRST superalgebra (6.4) we consider next the irreducible representations. Since (6.4) is the fermionic analogue of an inhomogeneous rotation group we anticipate that its representations will bear some resemblance to those of the Poincaré group. In Section 6.2 we find this to be the case. By consideration of the Casimirs of the group we label the eigenstates of the superalgebra with analogues of mass and spin. To complete the labelling of the states we consider two possibilities (i) momentum and (ii) helicity. However, when the momentum is used to complete the labelling of eigenstates, the states are unusual because they yield anticommuting eigenvalues for the fermionic operators and, if adopted as they stand, can give rise to nilpotent eigenvalues of observables which is physically nonsensical. We show in Appendix G with reference to the Grassmann oscillator that the correct way to circumvent this problem is to construct 'wave packets' over the idealised Grassmann states. This procedure is in principle no different from the situation in ordinary wave mechanics in which one obtains normalisable, physical wave functions by a superposition of monochromatic functions. We find that the Dirac notation may be consistently generalised to incorporate Grassmann states and operators. Having obtained these results in the appendix we set up and label the eigenstates of our BRST superalgebra within Section 6.2. In the next section we construct irreducible field representations for these basis states. In the final section, Section 6.4, we compare our work with other treatments of the BRST group.

With this analogy in mind we can express an irreducible finite dimensional representation of  $Sp(2)$  by a  $2s$  index multispinor where  $s = 0, \frac{1}{2}, 1, \dots$ . Then  $J_{kl}$  acts on this representation as

$$J_{kl} A_{m_1 \dots m_{2s}} = i \left( \sum_{j=1}^{2s} \eta_{km_j} A_{l m_1 \dots \overset{\wedge}{m_j} \dots m_{2s}} + \eta_{lm_j} A_{k m_1 \dots \overset{\wedge}{m_j} \dots m_{2s}} \right) \quad (6.11)$$

The indices  $m_i$  can only take the value 1 or 2. Clearly  $A_1 \dots_1$  has  $J_{12}$  eigenvalue  $-2is$  and  $A_2 \dots_2$  has  $J_{12}$  eigenvalue  $2is$ . For a general multispinor with  $\lambda_1$  indices of value 1 and  $\lambda_2$  indices of value 2 the  $J_{12}$  eigenvalue is  $-i(\lambda_1 - \lambda_2)$ . Hence  $\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)$  in (6.10).

Now from the commutation relations (6.4) we have

$$(J_{12} P_1 - P_1 J_{12}) |s; p^2; \lambda\rangle = -i P_1 |s; p^2; \lambda\rangle$$

hence

$$J_{12} P_1 |s; p^2; \lambda\rangle = -2i(\lambda + \frac{1}{2}) P_1 |s; p^2; \lambda\rangle$$

Similarly

$$J_{12} P_2 |s; p^2; \lambda\rangle = -2i(\lambda - \frac{1}{2}) P_2 |s; p^2; \lambda\rangle$$

We must also have

$$P^k P_k |s; p^2; \lambda\rangle = p^2 |s; p^2; \lambda\rangle$$

Combining these last three equations we obtain

$$P_1 |s; p^2; \lambda\rangle = p_1 |s; p^2; \lambda + \frac{1}{2}\rangle, P_2 |s; p^2; \lambda\rangle = p_2 |s; p^2; \lambda - \frac{1}{2}\rangle.$$

Thus the action of all the  $Sp(2)$  generators on the state is fixed.

and

$$\begin{aligned}
 W_{klm}\phi_n &= P_k J_{lm}\phi_n + (k,l,m) \\
 &= (i\partial_k)[-(\theta_l p_m + \theta_m p_l)A_n + i\eta_{ln}\phi_m + i\eta_{mn}\phi_l] + (k,l,m) \\
 &= -i(\eta_{lk}p_m + \eta_{mk}p_l)A_n + i\eta_{ln}p_k A_m(1-i\theta^r p_r) + i\eta_{mn}p_k A_l(1-i\theta^r p_r) + (k,l,m) \\
 &= i(\eta_{ln}p_k A_m + \eta_{mn}p_k A_l) - \eta_{ln}\theta_k A_m p^2/2 - \eta_{mn}\theta_k A_l p^2/2 + (k,l,m)
 \end{aligned}$$

Note for a scalar we would have  $W_{klm}\phi = 0$ .

Hence

$$\begin{aligned}
 W^{klm}W_{klm}\phi_n &= J^{kl}(i\partial^m)W_{klm}\phi_n + (k,l,m) \\
 &= -iJ^{kl}\partial^m(\eta_{ln}\theta_k A_m + \eta_{mn}\theta_k A_l + \eta_{mn}\theta_l A_k + \eta_{kn}\theta_l A_m + \eta_{kn}\theta_m A_l + \eta_{ln}\theta_m A_k)p^2/2 + (k,l,m) \\
 &= [i\Sigma^{kl}(-4\eta_{ln}A_k - 4\eta_{kn}A_l)p^2/2] + (k,l,m) \\
 &= [4(-6An)p^2/2] \times 3 \\
 &= -36 p^k p_k \phi_n
 \end{aligned} \tag{6.12}$$

In the general case  $\phi$  carries an arbitrary number of indices.

If we are looking for an irreducible representation then we can choose  $\phi$  to be symmetric in these indices. The requirement that  $\phi$  be an eigenfunction of  $P_m$  leads us, by analogy with the spinor case, to express  $\phi$  as

$$\phi_{m_1 \dots m_N} = A_{m_1 \dots m_N} \exp(-i\theta^k p_k)$$

Clearly  $P_l \phi_{m_1 \dots m_N} = p_l \phi_{m_1 \dots m_N}$  and it only remains to find the invariant

spin of the representation i.e. find  $w^2_\phi$ . We start with

$$\begin{aligned}
 W_{kln}\phi_{m_1 \dots m_N} &= (P_k J_{ln} + P_l J_{nk} + P_n J_{kl})A_{m_1 \dots m_N} \exp(-i\theta^r p_r) \\
 &= (-P_k(\theta_l p_n + \theta_n p_l) + P_k \Sigma_{ln} + (k,l,n))A_{m_1 \dots m_N} \exp(-i\theta^r p_r)
 \end{aligned}$$



### 6.3 FIELD REPRESENTATIONS

When working with gauge theories we need to express fields, rather than states, in representations of the appropriate group. For  $Sp(2) \wedge T_2$  we write the field as  $\Phi(\theta)$  where  $\theta$  is a Grassmann coordinate appended to ordinary space and time. Then the generators act as differential operators on the field.

$$P_k \Phi = (i\partial/\partial\theta^k)\Phi = i\partial_k \Phi$$

$$J_{mn} \Phi = (-i\theta_m \partial_n - i\theta_n \partial_m + \Sigma_{mn})\Phi$$

where the action of  $\Sigma$  on a multispinor is the same as the action of  $J$  in (6.11). It is then easy to set up the eigenfunctions corresponding to the bases (i) and (ii) given in Section 6.2.

(i) We consider as an example a spinor representation. Then by analogy with the spinor case we obtain a general representation for an eigenfunction of  $P_m$  and  $w^2$ .

A spinor carries one index. We can expand this as

$$\Phi_n = A_n + \theta^k B_{kn} + \frac{1}{2}\theta^2 C_n$$

Then  $P_1 \Phi_n = p_1 \Phi_n$  gives

$$P_1 \Phi_n = i(B_{1n} + \theta_1 C_n) = B_1 (A_n + \theta^k B_{kn} + \frac{1}{2}\theta^2 C_n)$$

This implies  $B_{1n} = -ip_1 A_n$  and  $C_n = \frac{1}{2}p^2 A_n$  so we can express  $\Phi$  as

$$\Phi_n = A_n \exp(-i\theta^k p_k)$$

Note for a scalar we would have obtained  $\Phi = A \exp(-i\theta^k p_k)$ .

In order to find  $w^2$  we first work out

$$\begin{aligned} J_{1m} \Phi_n &= (-i\theta_1 \partial_m - i\theta_m \partial_1 + \Sigma_{1m}) A_n (1 - i\theta^k p_k - \theta^2 p^2/4) \\ &= -(\theta_1 p_m + \theta_m p_1) A_n (1 - i\theta^k p_k) + i\eta_{1n} \Phi_n + i\eta_{mn} \Phi_1 \\ &= -(\theta_1 p_m + \theta_m p_1) A_n + i\eta_{1n} \Phi_m + i\eta_{mn} \Phi_1 \end{aligned}$$

$$\begin{aligned}
&= p_k [i \sum_{i=1}^N (\eta_{lm_i} A_{nm_1 \dots m_i \dots m_N}^{\wedge} + \eta_{nm_i} A_{lm_1 \dots m_i \dots m_N}^{\wedge})] \exp(-i\theta^r p_r) + (k, l, n) \\
&= ip_k \sum_{i=1}^N (\eta_{lm_i} A_{nm_1 \dots m_i \dots m_N}^{\wedge} + \eta_{nm_i} A_{lm_1 \dots m_i \dots m_N}^{\wedge}) \exp(-i\theta^r p_r) + (k, l, m)
\end{aligned}$$

Then we find

$$\begin{aligned}
w^{kl} w_{kl} \Phi_{m_1 \dots m_N} &= (p_k \sum l_n + p_l \sum n_k + p_n \sum k_l) w_{kl} \Phi_{m_1 \dots m_N} \\
&= [ip^k p_k \sum l_n (\sum_{i=1}^N \eta_{lm_i} A_{nm_1 \dots m_i \dots m_N}^{\wedge} + \eta_{nm_i} A_{lm_1 \dots m_i \dots m_N}^{\wedge}) \\
&\quad + ip^k p_l \sum l_n (\sum_{i=1}^N \eta_{nm_i} A_{km_1 \dots m_i \dots m_N}^{\wedge} + \eta_{km_i} A_{nm_1 \dots m_i \dots m_N}^{\wedge}) \\
&\quad + ip^k p_n \sum l_n (\sum_{i=1}^N \eta_{km_i} A_{lm_1 \dots m_i \dots m_N}^{\wedge} + \eta_{lm_i} A_{km_1 \dots m_i \dots m_N}^{\wedge})] \exp(-i\theta^r p_r) + (k, l, m) \\
&= -[3p^k p_k (\sum_{i \neq j} \eta_{lm_i} \delta_{mj}^n A_{nm_1 \dots m_i \dots m_j \dots m_N}^{\wedge} + \eta_{nm_i} \delta_{mj}^l A_{lm_1 \dots m_i \dots m_j \dots m_N}^{\wedge}) \\
&\quad + 3p^k p_k (\sum_{i=1}^N 4\eta_{lm_i} A_{m_1 \dots m_i \dots m_N}^{\wedge} + 2\eta_{lm_i} \delta_{m_i}^l A_{m_1 \dots m_i \dots m_N}^{\wedge}) \\
&\quad + 3p^l p_l (\sum_{i \neq j} \eta_{nm_i} \delta_{mj}^k A_{km_1 \dots m_i \dots m_j \dots m_N}^{\wedge} + \eta_{lm_i} \delta_{mj}^n A_{nm_1 \dots m_i \dots m_j \dots m_N}^{\wedge}) \\
&\quad + 3p^l p_l (\sum_{i=1}^N 4\eta_{nm_i} A_{m_1 \dots m_i \dots m_N}^{\wedge} + 2\eta_{nm_i} \delta_{m_i}^k A_{m_1 \dots m_i \dots m_N}^{\wedge})] \exp(-i\theta^r p_r) \\
&\quad \text{(where we have used } A_{lm_1 \dots m_N}^{\wedge} = 0 \text{ and } p^k p_l = \frac{1}{2} \delta_{kl} p^2) \\
&= (12p^k p_k \binom{N}{2} + 36p^k p_k \binom{N}{1}) A_{m_1 \dots m_N} \exp(-i\theta^r p_r) \\
&= 12N(N+2) p^k p_k \Phi_{m_1 \dots m_N}
\end{aligned}$$

For  $N = 1$  this reduces to (6.12) as it should. Writing  $N = 2s$  where  $s$  is the spin we find, for  $w^2 = w^{kl} w_{kl}/48$ ,

$$w^2 = p^2 s(s+1)$$

which was mentioned in Section 6.2.

These results give the representation functions in the  $|j;p\rangle$  basis as

$$\phi_{m_1 \dots m_{2s}}(\theta) = \{m_1, \dots, m_{2s}, \theta | s; p\} = A_{m_1 \dots m_{2s}} \exp(-i\theta^k p_k)$$

(ii) We now find a field representation for eigenfunction of  $p^2$  and  $J_{12}$  corresponding to the second choice of basis in Section 6.2.

First we expand our field as

$$\phi_{m_1 \dots m_{2s}} = a_{m_1 \dots m_{2s}} + \theta^k b_{km_1 \dots m_{2s}} + \frac{1}{2} \theta^2 c_{m_1 \dots m_{2s}}$$

then  $p^2 \phi_{m_1 \dots m_{2s}} = p^2 a_{m_1 \dots m_{2s}}$  gives us

$$2c_{m_1 \dots m_{2s}} = p^2 (a_{m_1 \dots m_{2s}} + \theta^k b_{km_1 \dots m_{2s}} + \frac{1}{2} \theta^2 c_{m_1 \dots m_{2s}})$$

hence  $p^2 a_{m_1 \dots m_{2s}} = 2c_{m_1 \dots m_{2s}}$  and  $p^2 b_{m_1 \dots m_{2s}} = 0$ . Now if we also

require the field to be an eigenfunction of  $J_{12}$  with eigenvalue  $-2i\lambda$  then first we work out

$$J_{k1} \phi_{m_1 \dots m_{2s}} = -i\theta^k b_{1m_1 \dots m_{2s}} - i\theta^1 b_{km_1 \dots m_{2s}} + \Sigma_{k1} (a_{m_1 \dots m_{2s}} + \theta^k b_{km_1 \dots m_{2s}} + \frac{1}{2} \theta^2 c_{m_1 \dots m_{2s}})$$

hence

$$\begin{aligned} J_{12} \phi_{m_1 \dots m_{2s}} &= -i\theta^1 b_{2m_1 \dots m_{2s}} - i\theta^2 b_{1m_1 \dots m_{2s}} + \Sigma_{12} (a_{m_1 \dots m_{2s}} + \theta^k b_{km_1 \dots m_{2s}} + \frac{1}{2} \theta^2 c_{m_1 \dots m_{2s}}) \\ &= -2i\lambda (a_{m_1 \dots m_{2s}} + \theta^k b_{km_1 \dots m_{2s}} + \frac{1}{2} \theta^2 c_{m_1 \dots m_{2s}}) \end{aligned}$$

Then we obtain

$$\Sigma_{12} a_{m_1 \dots m_{2s}} = -2i\lambda a_{m_1 \dots m_{2s}}$$

$$\Sigma_{12} b_1 m_1 \dots m_{2s} = -2i(\lambda + \frac{1}{2}) b_1 m_1 \dots m_{2s}$$

$$\Sigma_{12} b_2 m_1 \dots m_{2s} = -2i(\lambda - \frac{1}{2}) b_2 m_1 \dots m_{2s}$$

These imply that we can write our eigenfunction in this basis as

$$\begin{aligned} \Phi_{m_1 \dots m_{2s}} &= \{m_1 \dots m_{2s}, \theta | p^2; s; \lambda\} \\ &= a_{m_1 \dots m_{2s}} + \theta^k a_{k m_1 \dots m_{2s}} + \frac{1}{2} \theta^2 p^2 a_{m_1 \dots m_{2s}} \end{aligned}$$

#### 6.4 DISCUSSION

In this section we compare our work with earlier important references on the subject.

Kugo and Ojima [7] based their canonical, covariant proof of physical S-matrix unitarity on an irreducible representation of the algebra of  $Q_B$  and  $Q_C$ . They found two possibilities (i) singlets and (ii) doublets. A state  $|\alpha\rangle$  is a singlet if  $Q_B |\alpha\rangle = 0$  and  $|\alpha\rangle \neq Q_B |\beta\rangle$  for any state  $|\beta\rangle$ . If  $Q_B |\alpha\rangle \neq 0$  then the states  $|\alpha\rangle$  and  $Q_B |\alpha\rangle$  form a doublet. Doublet states always occur in pairs forming quartets. The members of quartets then appear only in zero-norm combinations. Labelling the states by the eigenvalue  $k$  of  $iQ_C$  we make a distinction between singlets with  $k = 0$  which are physical particles and singlets with  $k \neq 0$  which can violate unitarity. It was assumed by Kugo and Ojima, and later proved as part of a theorem by Nakanishi [8], that singlet states with  $k \neq 0$  do not exist. Note here that  $iQ_C$  must have integral eigenvalues by its identification as ghost number but these are not imposed by the algebra.

Bonora et al [9] then found the irreducible representations of the extended BRST group. They introduced the term null to describe a state which may be expressed as a linear combination of  $Q_B |\beta\rangle$  and  $Q_B |\gamma\rangle$  for some states  $|\beta\rangle, |\gamma\rangle$ . Then they were able to prove very

simply that any physical state with  $k \neq 0$  is a null state which is analogous to the result that singlets with  $k \neq 0$  do not exist by Nakanishi [8]. A physical state here satisfies  $0 = Q_B |\alpha\rangle = \bar{Q}_B |\alpha\rangle$ . The proof required the introduction of a non-conserved charge  $R$  which satisfies

$$iQ_C = \{Q_B, \bar{R}\} + \{Q_B, R\}.$$

Then for a physical state  $|\alpha, k\rangle$  with  $k \neq 0$  we have  $iQ_C |\alpha, k\rangle = k |\alpha, k\rangle = \{Q_B, \bar{R}\} |\alpha, k\rangle + \{Q_B, R\} |\alpha, k\rangle$

hence

$$|\alpha, k\rangle = [Q_B(\bar{R}|\alpha, k\rangle) + \bar{Q}_B(R|\alpha, k\rangle)]/k$$

so  $|\alpha, k\rangle$  is null by definition. However in addition to singlet and quartet representations Bonora and Tonin also found chains of finite or infinite length. It was then necessary to resort to Nakanishi's full theorem to exclude the finite chain possibility. The infinite chain representation remained as a possibility.

The next step was made by Nishijima [5]. It had previously been found by Nakanishi and Ojima [4] that gauge theories in the Landau gauge admit a larger symmetry group. Nishijima added two of the generators,  $Q$  and  $\bar{Q}$  to the extended BRST algebra. Then  $Q, \bar{Q}$  which are only conserved in the Landau gauge and  $iQ_C$  form an algebra isomorphic to  $SU(2)$ . The algebra then imposes the condition that  $iQ_C$  have integral eigenvalues. Also for the matrix representations of  $Q, \bar{Q}$  to be well defined we must have a finite dimensional representation. This excluded the infinite chain possibility.

In our work the charges  $Q = J_{11}$  and  $\bar{Q} = J_{22}$  are conserved for all covariant  $Sp(2)$  invariant gauge fixing terms not just the Landau gauge.

$iQ_c = J_{12}$  has integral eigenvalues which we have written as  $2i\lambda$  previously. Our choice of a finite dimensional representation was used to obtain the consistent reality conditions in (7.5). Thus we have also excluded any infinite chain representations. Nishijima's quartet states simply correspond to the basis vectors of Section 6.2 part (ii). Our work has stressed the importance of the Casimirs of the extended group and the usefulness of Grassmann vectors.

Finally we should consider the situation when the supertranslation group is trivially represented,  $P \rightarrow 0$ . Then both of the Casimirs,  $p^2$  and  $w^2$ , disappear. This is an important case since all physical states are BRST invariant. In this case we should focus on the  $Sp(2)$  group and its Casimir which roughly speaking is the ratio of  $w^2$  to  $p^2$  and non-zero in general. Genuine physical particles will have  $\lambda = 0$ . For  $\lambda \neq 0$  we can repeat the existence of R argument of Bonora and Tonin and find that such states are null.

The results obtained in this chapter have been published in [10]. This completes our study of the Grassmann Euclidean group and the main work of our thesis. In the next chapter we conclude the thesis with a summary of the work done and a survey of the possible future developments.

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## 7 CONCLUSIONS AND PROSPECTS

In this final chapter we summarise the original results presented in this thesis and discuss these with reference to prospects for future research.

The CSDR scheme has enjoyed some success with model builders because it gives a reduction in the number of arbitrary parameters in Yang-Mills-Higgs theories. The parameters of the four-dimensional theory are related to the higher-dimensional coupling constant and the size of the coset space. However these relationships are only predicted at the classical level. In Chapter 3 we presented our first original results. Applying the background-field formalism of Jack and Osborn [1] to a wide variety of models we found that the relationships predicted by CSDR were untenable at one-loop order. We conclude that the CSDR scheme, as it stands, has no predictive power beyond the classical level.

If the CSDR scheme is to give a true reduction in the number of arbitrary parameters then the relationships must be consistently renormalisable. Possible solutions to this are to impose additional symmetries on the model or to seek more general forms for the higher-dimensional action. In models preserving (conventional) supersymmetry the scalar potential receives no quantum corrections so the CSDR relationships are automatically preserved.

Going beyond the problem of obtaining a consistently renormalisable model the CSDR scheme needs to be studied and extended in several ways. For example, we need to look more closely at the connection between the finite modes selected by CSDR and the massless modes of the harmonic expansion. That these are not identical has been shown by Palla [2]. In addition the formalism needs to be extended to the (conventional)



supersymmetric case if CSDR is to have any role to play in the dimensional reduction of supergravity and superstring theories. The authors in [3] have made some progress in this direction. Finally, we need to study a wider range of coset spaces and in particular non-symmetric ones [4].

The orthosymplectic superalgebra offers a simple but lucid way of obtaining the correct ghost and gauge fields [5]. It is strongly connected to the BRST symmetry and in this context has many avenues open to it. Only recently it has found an application in string theory [6] following the study of BRST quantisation of strings.

In [7] the authors applied the CSDR scheme to vector gauge fields in a representation of  $OSp(4/2)$  over a coset space  $Sp(2) \wedge T_2$ . After dimensional reduction the theory was BRST invariant and had the correct ghost spectrum. In Chapter 4 we extended this formalism for the first time, to higher-rank, antisymmetric tensor fields. After dimensional reduction we obtained a ghost-spectrum and degree-of-freedom count in agreement with the BRST quantisation performed by other authors using different methods.

In Chapter 5 we extended the formalism, once more, to spinor-vector gauge fields. In this case it was necessary to take our fields in an infinite dimensional representation of  $OSp(4/2)$  since finite, spinor representations have zero superdimension and hence zero degrees of freedom. In order for our dimensionally reduced theory to include the massless, Rarita-Schwinger field it was necessary to reduce our coset space from  $Sp(2) \wedge T_2 / Sp(2)$  to  $U(1) \wedge T_2 / U(1)$ . Note this coset space could have been used in [6] and Chapter 4 without altering the results. For a certain choice of gauge-fixing parameters our action reduced to the spinor sector of supergravity [8] with the usual spectrum of Fadeev-Popov and Nielsen-Kallosh ghosts. However our gauge has not

been completely fixed by the addition of the usual superfield gauge-fixing terms. The solution to this problem may lie in the study of more exotic gauge-fixing terms. This would be an unappealing scenario for a method whose appeal, to date, lies in its simplicity. Alternatively, the solution may be found, at a more fundamental level, in a different choice of conjugation properties for our  $\theta^m$  [9].

In our program of applying the CSDR method to BRST quantisation of fields there are still some problems open to further investigation. For example we could extend our methods to fields in non-Abelian groups. This was done in [7] for the vector gauge fields. It would be particularly interesting in our case for the spinor-vector field. In addition we could extend the formalism to the gravitational field.

The grassmann euclidean group  $Sp(2) \wedge T_2$  underlay all of our applications of CSDR to BRST quantisation. However it is interesting in its own right. In Chapter 6 we constructed, for the first time, the irreducible representations of this group. In analogy to the Poincaré group the state vectors were labelled by spin and grassmann momentum. The latter if adopted as a label for a physical state vector would give nilpotent numbers as observables. Instead the physical state vectors were taken as wave packets over the momentum. It was found possible to generalise Dirac's notation to grassmann operators and states.

Finally, it would be desirable to have a superspace equivalent of many of our four-dimensional constructs. It would be interesting, for example, to investigate whether equations of motion and propagators are invariant under the dimensional reduction procedure. Work on superfield equations of motion has commenced but the presence of  $x^2$  in our higher-dimensional actions is an impediment to the construction of super-propagators as there is no analogue of this in ordinary four-dimensional actions.

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## APPENDIX A: NOTATION AND CONVENTIONS

Our coordinates are  $x^M = (x^\mu, \theta^m)$  with  $\theta^m = (\theta^5, \theta^6) = (\theta, \bar{\theta})$ .

Our metric is given by

$$\eta^{MN} = \begin{pmatrix} \eta^{\mu\rho} & \\ & \eta^{mn} \end{pmatrix}$$

where  $\eta^{\mu\rho}$  is the usual Lorentz, diagonal metric and

$$\eta^{mn} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The metric is used to raise and lower indices from the left and contracting inwards

$$\eta^{MN} T \dots_N \dots = T \dots^M \dots, \eta_{MN} T \dots^N \dots = T \dots_M \dots$$

It is convenient to use a sign factor  $[MN]$  with

$$[\mu\rho] = 1 = [\mu m]$$

$$[mn] = -1.$$

$$\text{Thus } \eta^{MN} = [MN] \eta^{NM} \text{ and } x^M x^N = [MN] x^N x^M.$$

The following identities are useful in the text

$$\eta_{mn} \eta^{pq} = \delta_n^p \delta_m^q - \delta_m^p \delta_n^q \quad (\text{A.1})$$

$$\eta_{mn} \eta_{pq} = \eta_{mp} \eta_{nq} - \eta_{mq} \eta_{np}$$

$$\theta^p \theta^q = \frac{1}{2} \theta^2 \epsilon^{qp}$$

$$\partial^q \theta^p = \epsilon^{qp}$$

$$\partial^q \theta_q = -2 = -\partial_q \theta^q$$

Our Dirac matrices obey

$$\{\gamma_\mu, \gamma_\rho\} = 2\eta_{\mu\rho}, \gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu, \gamma^0 \gamma^5 \gamma^0 = -\gamma^5, \gamma^5 \gamma_5 = 1$$

## APPENDIX B: CALCULUS OF SU(r) INVARIANT TENSORS

Application of (3.10) to the scalar potential (3.49) of case IV involves the calculus of the SU(r) invariant tensors  $f_{abc}$  and  $d_{abc}$ . We quote here some of the intermediate formulae in the calculation. Extensive compilations of such formulae are available in the literature. For example see [12].

The basic relationship is

$$\lambda_a \lambda_b = i f_{abc} \lambda_c + d_{abc} \lambda_c + 2\delta_{ab}/r \quad (B.1)$$

Then for example

$$\text{tr}(\lambda_a \lambda_b \lambda_c) = 2i f_{abc} + 2d_{abc} \quad (B.2)$$

$$d_{aef} d_{bef} = (r^2 - 4)\delta_{ab}/r \quad (B.3)$$

Other required products can be extracted by reducing expressions to  $\lambda$  traces and using the completeness relation in the forms

$$\frac{1}{2} \lambda_a \chi \lambda_a = (\text{tr} \chi) \cdot 1 - \chi/r \quad (B.4)$$

$$\frac{1}{2} \text{tr}(\chi \lambda_a) \text{tr}(\lambda_a \kappa) = \text{tr}(\chi \kappa) - \text{tr}(\chi) \text{tr}(\kappa)/r \quad (B.5)$$

Thus for example

$$\text{tr}(\lambda_a \lambda_e \lambda_a \lambda_f) \phi_e \phi_f = -4\phi^2/r \quad (B.6)$$

$$\text{tr}(\lambda_a \lambda_e \lambda_b \lambda_f) \phi_a \phi_b \phi_e \phi_f = 4(\phi^2)^2/r + 2(\phi_* \phi)^2 \quad (B.7)$$

$$\text{tr}(\lambda_a \lambda_e \lambda_b \lambda_f) \phi_e \phi_f d_{abc} \chi_c = -8d_{abc} \phi_a \phi_b \chi_c/r \quad (B.8)$$

$$\text{tr}(\lambda_e \lambda_a \lambda_f \lambda_b) \text{tr}(\lambda_e \lambda_c \lambda_f \lambda_d) \phi_a \phi_b \phi_c \phi_d = 16(r^2 - 1)(\phi^2)^2/r^2 - 16(\phi_* \phi)^2/r \quad (B.9)$$

are all needed either in  $\text{tr}(U'^2)$  or  $\text{tr}(P^2)$ .

APPENDIX C :  $\Sigma^{mn} \Sigma_{mn}$ 

In this appendix we find the action of  $C_2 = \Sigma^{mn} \Sigma_{mn}$  on a tensor with  $r$  odd indices.

We have

$$(C_2)_{p_1 \dots p_r}^{s_1 \dots s_r} = (\Sigma^{mn})_{p_1 \dots p_r}^{k_1 \dots k_r} (\Sigma_{mn})_{k_1 \dots k_r}^{s_1 \dots s_r}$$

Now the action of  $\Sigma_{mn}$  is given by

$$(\Sigma_{mn})_{k_1 \dots k_r}^{s_1 \dots s_r} T_{s_1 \dots s_r} = \sum_{i=1}^r (\eta_{mk_i}^{s_i} \delta_n^{s_i} + \eta_{nk_i}^{s_i} \delta_m^{s_i}) \Pi_{j \neq i}^r \delta_{k_j}^{s_j} T_{s_1 \dots s_r}$$

and the action of  $\Sigma^{mn}$  by

$$(\Sigma^{mn})_{p_1 \dots p_r}^{k_1 \dots k_r} T_{k_1 \dots k_r} = \sum_{i=1}^r (\eta^{mk_i}_{\delta p_i} + \eta^{nk_i}_{\delta p_i}) \Pi_{j \neq i}^r \delta_{p_j}^{k_j} T_{k_1 \dots k_r}$$

So we have

$$\begin{aligned} (C_2)_{p_1 \dots p_r}^{s_1 \dots s_r} &= \sum_{i,l=1}^r (-\delta_{k_l}^{s_l} \delta_{p_i}^{k_i} + \eta_{p_i k_l}^{s_l} \delta_{p_i}^{s_l} + \eta_{p_i k_l}^{s_l} \delta_{p_i}^{s_l} + \eta_{p_i k_l}^{s_l} \delta_{p_i}^{s_l}) \Pi_{j \neq i, m \neq l}^r \delta_{p_j}^{k_j} \delta_{k_m}^{s_m} \\ &= \sum_{i,l=1}^r (-4\delta_{k_l}^{s_l} \delta_{p_i}^{k_i} + 2\delta_{k_l}^{s_l} \delta_{p_i}^{k_i}) \Pi_{j \neq i, m \neq l}^r \delta_{p_j}^{k_j} \delta_{k_m}^{s_m} \end{aligned}$$

(where we have used  $\eta^{ab} \eta_{cd} = \delta_c^b \delta_d^a - \delta_d^b \delta_c^a$ )

$$\begin{aligned} &= \sum_{i=1}^r -6\delta_{p_i}^{s_i} \Pi_{j \neq i}^r \delta_{p_j}^{s_j} \\ &+ \sum_{i \neq l}^r (-4\delta_{p_l}^{s_l} \delta_{p_i}^{s_l} + 2\delta_{p_l}^{s_l} \delta_{p_i}^{s_l}) \Pi_{j \neq i, l}^r \delta_{p_j}^{s_j} \\ &= 2r(r-4) \Pi_{i=1}^r \delta_{p_i}^{s_i} - 4 \sum_{i \neq l}^r \delta_{p_l}^{s_l} \delta_{p_i}^{s_l} \Pi_{j \neq i, l}^r \delta_{p_j}^{s_j} \end{aligned} \quad (C.1)$$

Now suppose  $T_{s_1 \dots s_r}$  has the symmetry properties

$$T_{s_1 s_2 s_3 \dots s_r} = T_{s_2 s_1 s_3 \dots s_r}$$

$$T_{s_1 s_2 s_3 \dots s_i \dots s_j \dots s_r} = T_{s_1 s_2 s_3 \dots s_j \dots s_i \dots s_r} \quad \text{for } i, j = 3 \text{ to } r$$

In fact this is the only sort of tensor we will require to act  $C_2$  on.

For convenience, letting  $r = q + 2$ , we write (C.1) as

$$\begin{aligned} (C_2)_{kl} p_1 \dots p_q & \quad mns_1 \dots s_q T_{mns_1 \dots s_q} = (2(q+2)(q-2) - 4(2) - 4q(q-1)) T_{kl} p_1 \dots p_q \\ & - 8 \sum_{i=1}^q (T_{p_i l k p_1 \dots \hat{p}_i \dots p_q} + T_{kp_i l p_1 \dots \hat{p}_i \dots p_q}) \\ & = -2(q^2 - 2q + 8) T_{kl} p_1 \dots p_q - 8 \sum_{i=1}^q (T_{p_i l k p_1 \dots \hat{p}_i \dots p_q} + T_{kp_i l p_1 \dots \hat{p}_i \dots p_q}) \end{aligned} \quad (C.2)$$

## APPENDIX D : FORMULAE FOR GAUGE PARAMETER CONSTRAINT

In this Appendix we solve the constraint (4.49). We assume  $q > 1$  since  $q = 1$  is a special case and has been solved in Section 4.2.

First we decompose  $A_{k|M_1 \dots M_R}$  into a sum of three terms

$$A_{k|M_1 \dots M_R} = \alpha_1 A^{<q+2>}_{k|M_1 \dots M_R} + \alpha_2 A^{<q>}_{k|M_1 \dots M_R} + \alpha_3 A^{<q-2>}_{k|M_1 \dots M_R} \quad (D.1)$$

where

$$A^{<q+2>}_{k|M_1 \dots M_R} = A_{k|M_1 \dots M_R} + \sum_{i=1}^q (A_{kM_i | M_1 \dots \hat{M}_i \dots M_R} + A_{M_i | kM_1 \dots \hat{M}_i \dots M_R}) \\ + \sum_{1 \leq i < j \leq q} A_{M_i M_j | kM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}$$

$$A^{<q>}_{k|M_1 \dots M_R} = \sum_{kl} \sum_{i=1}^q A_{rM_i | M_1 \dots \hat{M}_i \dots M_R}$$

$$A^{<q-2>}_{k|M_1 \dots M_R} = \sum_{i=1}^q \sum_{j=i}^q \eta_{kM_i} \eta_{M_j} A^{rs}_{rsM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}$$

It is not obvious what the  $\alpha_i$  are. To find them we observe that there are  $\binom{q+2}{2} = \frac{1}{2}(q+2)(q+1)$  distinct terms which can be obtained by the interchange of odd indices in  $A_{k|M_1 \dots M_R}$ . That there are only  $\binom{q+2}{2}$  and not  $q!$  such terms is due to the symmetry properties

$$A_{k|M_1 \dots M_R} = A_{l|M_1 \dots M_R}$$

$$A_{k|M_1 \dots M_i \dots M_j \dots M_R} = A_{k|M_1 \dots M_j \dots M_i \dots M_R} \quad \text{for } i, j=1 \text{ to } q.$$

The distinct terms are  $A_{k|M_1 \dots M_R}$ ,  $A_{kM_i | M_1 \dots \hat{M}_i \dots M_R}$ ,  $A_{M_i | kM_1 \dots \hat{M}_i \dots M_R}$

and  $A_{M_i M_j | kM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}$  for  $1 \leq i < j \leq q$ .

We can then express the  $A^{<r>}_{k|M_1 \dots M_R}$  for  $r = q, q-2$  as

$$A^{<q>}_{k|M_1 \dots M_R} = (-2q)A_{k|M_1 \dots M_R} + (2-q) \sum_{i=1}^q A_{M_i | kM_1 \dots \hat{M}_i \dots M_R} \\ + (2-q) \sum_{i=1}^q A_{kM_i | M_1 \dots \hat{M}_i \dots M_R} + 4 \sum_{1 \leq i < j \leq q} A_{M_i M_j | kM_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}$$



$$A^{<q-2>}_{klM_1 \dots M_R} = q(q-1)A_{klM_1 \dots M_R} - (q-1) \sum_{i=1}^q A_{M_i l k M_1 \dots \hat{M}_i \dots M_R} \\ - (q-1) \sum_{i=1}^q A_{k M_i l M_1 \dots \hat{M}_i \dots M_R} + 2 \sum_{1 \leq i < j \leq q} A_{M_i M_j k l M_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R} \quad (D.2)$$

$A^{<q+2>}_{klM_1 \dots M_R}$  need not be rewritten. From these expressions we obtain the following equations for the  $\alpha_i$

$$1 = \alpha_1 - 2q\alpha_2 + q(q-1)\alpha_3$$

$$0 = \alpha_1 + (2-q)\alpha_2 - (q-1)\alpha_3$$

$$0 = \alpha_1 + 4\alpha_2 + 2\alpha_3$$

Solving these we obtain

$$\alpha_1 = 2/(q+1)(q+2)$$

$$\alpha_2 = -1/q(q+2)$$

$$\alpha_3 = 1/q(q+1) \quad (D.3)$$

Now the constraint is

$$\sum_{kl} A_{mnM_1 \dots M_R} - \sum_{mn} A_{klM_1 \dots M_R} = \eta_{km} A_{lnM_1 \dots M_R} + \eta_{kn} A_{mlM_1 \dots M_R} + \eta_{lm} A_{knM_1 \dots M_R} \\ + \eta_{ln} A_{mkM_1 \dots M_R} :$$

Multiplying through by  $\sum^{mn}$  we can write this as

$$C_2 A_{klM_1 \dots M_R} = \sum^{mn} \sum_{i=1}^q (\eta_{km} A_{mn l M_1 \dots \hat{M}_i \dots M_R} + \eta_{ln} A_{m k M_1 \dots \hat{M}_i \dots M_R}) \quad (D.4)$$

In order to solve this we express both sides first as a combination of the  $\binom{q+2}{2}$  distinct terms then as a combination of the  $A^{<r>}_{mnM_1 \dots M_R}$

for  $r = q, q+2, q-2$ . Equating both sides we then determine which of the  $A^{<r>}_{mnM_1 \dots M_R}$  survive the constraint.

Starting with the left-hand side of (D.4) we have from Appendix C equation (C.2)

$$C_2 A_{klM_1 \dots M_R} = -2(q^2 - 2q + 8) A_{klM_1 \dots M_R}^{-8} \sum_{i=1}^q (A_{M_i l k M_1 \dots \hat{M}_i \dots M_R} + A_{M_i k l M_1 \dots \hat{M}_i \dots M_R}) \quad (D.5)$$

Then if we write

$$C_2 A_{klM_1 \dots M_R} = \alpha_1^{<q+2>} A_{klM_1 \dots M_R}^{<q+2>} + \alpha_2^{<q>} A_{klM_1 \dots M_R}^{<q>} + \alpha_3^{<q-2>} A_{klM_1 \dots M_R}^{<q-2>} \quad (D.6)$$

where  $\alpha_i^{<r>} = \alpha_i C_2^{<r>}$  for  $i=1,2,3$  and  $r=q+2, q, q-2$  we obtain by comparison with (D.2) and (D.3).

$$-2(q^2 - 2q + 8) = \alpha_1^{<q+2>} - 2q\alpha_2^{<q>} + q(q-1)\alpha_3^{<q-2>}$$

$$-8 = \alpha_1^{<q+2>} + (2-q)\alpha_2^{<q>} - (q-1)\alpha_3^{<q-2>}$$

$$0 = \alpha_1^{<q+2>} + 4\alpha_2^{<q>} + 2\alpha_3^{<q+2>}$$

The solutions are

$$\alpha_1^{<q+2>} = -4(q+4)/(q+1)$$

$$\alpha_2^{<q>} = 2$$

$$\alpha_3^{<q-2>} = -2(q-2)/(q+1)$$

These imply that  $C_2^{<r>} = -2r(r+2)$  for  $r \geq 0$ .

Having expressed the left-hand side of (D.4) in the desired form we can similarly express the right-hand side in terms of  $A_{klM_1 \dots M_R}^{<r>}$  by

$$\begin{aligned} & \sum_{i=1}^{mn} \sum_{j=1}^q (\eta_{kM_i} A_{mn l M_1 \dots \hat{M}_i \dots M_R} + \eta_{lM_i} A_{mn k M_1 \dots \hat{M}_i \dots M_R}) \\ &= \sum_{i=1}^q \sum_{j \neq i}^q (2\eta_{kM_j} \eta^{nm} A_{M_i n m l M_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R} + 2\eta_{lM_j} \eta^{nm} A_{M_i n m k M_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R}) \\ &+ \sum_{i=1}^q (2\eta_{kM_i} \eta^{nm} A_{l n m M_1 \dots \hat{M}_i \dots M_R} + 2\eta_{lM_i} \eta^{nm} A_{k n m M_1 \dots \hat{M}_i \dots M_R}) \\ &= \sum_{i=1}^q (\sum_{j \neq i}^q 4A_{M_i M_j k l M_1 \dots \hat{M}_i \dots \hat{M}_j \dots M_R} - 2A_{k M_i l M_1 \dots \hat{M}_i \dots M_R} - 2A_{l M_i k M_1 \dots \hat{M}_i \dots M_R}) \\ &+ \sum_{i=1}^q (2A_{k M_i l M_1 \dots \hat{M}_i \dots M_R} + 2A_{l M_i k M_1 \dots \hat{M}_i \dots M_R}) - 4q A_{kl M_1 \dots M_R} \end{aligned}$$

where we have used (A.1).

From (D.2) this becomes

$$\Sigma^{mn} \Sigma_{kl} A_{mnM_1 \dots M_R} = -a A^{<q+2>}_{klM_1 \dots M_R} - b A^{<q>}_{klM_1 \dots M_R} - c A^{<q-2>}_{klM_1 \dots M_R} \quad (D.7)$$

where  $a, b, c$  satisfy

$$4q = a - 2qb + q(q-1)c$$

$$2q = a + (2-q)b - (q-1)c + 4$$

$$-8 = a + 4b + 2c$$

We find  $a = c = 0, b = 2$ .

Now we can equate both sides of (D.4) expressed in terms of the

$A^{<r>}_{klM_1 \dots M_R}$ . We obtain

$$-4(q+4) A^{<q+2>}_{klM_1 \dots M_R} = 0$$

$$2A^{<q>}_{klM_1 \dots M_R} = 2A^{<q>}_{klM_1 \dots M_R}$$

$$-2(q-2)A^{<q-2>}_{klM_1 \dots M_R} = 0$$

From this the only survivors of the constraint for  $q > 1$  are

$A^{<q>}_{klM_1 \dots M_R}$  and  $A^{<q-2>=<0>}_{klM_1 \dots M_R}$ . Note this is also true for

$q=1$  as proved in Section 4.2.

APPENDIX E : INVARIANCE OF  $\bar{\Psi}\Psi$ 

In this Appendix we confirm our statement that  $\bar{\Psi}\Psi$  is invariant under a transformation  $\Psi \rightarrow \Psi + X\Psi$  if  $X$  is anti-hermitian i.e.  $\gamma_0 X^\dagger \gamma_0 = -X$  and the adjoint is defined by  $\bar{\Psi} = \Psi^\dagger \gamma_0$  in an infinite dimensional representation. We then check explicitly that  $X$  is anti-hermitian for the  $OSp(4/2)$  transformation of interest in our work.

First, let us expand  $\Psi$  as a sum over basis states showing the spinor index,  $\alpha$ , explicitly.

$$\Psi_\alpha = \sum_i |i\rangle \eta_{ij} \langle j | \Psi_\alpha \rangle$$

Under an infinitesimal transformation  $\Psi \rightarrow \Psi + X\Psi$  we have

$$\delta \Psi_\alpha = \sum_{ij} X_\alpha^\beta |i\rangle \eta_{ij} \langle j | \Psi_\beta \rangle$$

Defining  $\bar{\Psi} = \Psi^\dagger \gamma_0$  we can write  $\bar{\Psi}$  as a sum over basis states as

$$\bar{\Psi}^\alpha = \sum_{ij} \langle i | \bar{\Psi}^\dagger \gamma_0^\beta | j \rangle \eta_{ij} \gamma_{0\beta}^\alpha$$

and the variation in  $\bar{\Psi}$  is given by  $\delta \bar{\Psi} = \delta \Psi^\dagger \gamma_0$  hence

$$\delta \bar{\Psi}^\alpha = \sum_{ij} \langle i | \delta \Psi^\dagger \gamma_0^\beta | j \rangle \eta_{ij} \gamma_{0\beta}^\alpha$$

Then  $\delta(\bar{\Psi}\Psi)$  is given by

$$\begin{aligned} \delta(\bar{\Psi}^\alpha \Psi_\alpha) &= \delta \bar{\Psi}^\alpha \Psi_\alpha + \bar{\Psi}^\alpha \delta \Psi_\alpha \\ &= \sum_{ijk} \langle i | \delta \Psi^\dagger \gamma_0^\beta | j \rangle \eta_{ij} \gamma_{0\beta}^\alpha \langle k | \Psi_\alpha \rangle \\ &\quad + \sum_{ijk} \langle i | \bar{\Psi}^\dagger \gamma_0^\beta | j \rangle \eta_{ij} \gamma_{0\beta}^\alpha X_\alpha^\sigma \langle k | \Psi_\sigma \rangle \\ &= \sum_{ijk} \langle i | \delta \Psi^\dagger \gamma_0^\beta | j \rangle \eta_{ij} \gamma_{0\beta}^\alpha (X_\alpha^\sigma \gamma_{0\sigma}^\alpha + \gamma_{0\beta}^\sigma X_\sigma^\alpha) \langle k | \Psi_\alpha \rangle \end{aligned}$$

If  $X$  satisfies  $\gamma_0 X^\dagger \gamma_0 = -X$  then  $\delta(\bar{\Psi}\Psi) = 0$  as stated.

We now check that  $X$  is anti-hermitian for  $OSp(4/2)$  transformations.

Under a super Lorentz transformation the coordinates transform as given in (5.8)

$$x^M \rightarrow x^M + x^N \lambda_N^M \quad (E.1)$$

The spinors  $\Psi$  transform as in (5.)

$$\Psi \rightarrow \Psi - \frac{1}{2} \lambda^{KL} \Sigma_{LK} \Psi$$

where  $\Sigma_{KL} = -(\Gamma_K \Gamma_L - [KL] \Gamma_L \Gamma_K)/4$ .

If  $\bar{\Psi}\Psi$  is to be invariant under  $OSp(4/2)$  transformations then we require  $\gamma_0 (\lambda^{KL} \Sigma_{LK})^\dagger \gamma_0 = -\lambda^{KL} \Sigma_{LK}$ . Now our coordinates obey

$$x^{M\dagger} = x^N c_N^M \quad (E.2)$$

where  $c_\mu^\nu = \delta_\mu^\nu$  (since  $x^{\mu\dagger} = x^\mu$ ) and  $c_n^m = \begin{pmatrix} 01 \\ 10 \end{pmatrix}$  (since  $\theta^m = (\theta^5, \theta^6) = (\theta, \bar{\theta})$ ) (E.1) and (E.2) imply

$$\lambda^{KL\dagger} = [K][KL] c^{KP} \lambda_P^N c_N^L$$

Hence  $\bar{\Psi}\Psi$  is invariant provided

$$\Gamma_N^\dagger \gamma_0 = [N] \gamma_0 c_N^M \Gamma_M$$

if we assume  $\Gamma_\mu^\dagger \gamma_0 = \gamma_0 \Gamma_\mu$ .

APPENDIX F : ACTION OF  $\Gamma^M$  ON BASIS STATES

In this appendix we find the action of the generalised Dirac matrices,  $\Gamma^M$ , on the infinite dimensional set of basis states  $|i\rangle$ .

We choose

$$\Gamma^M = (\gamma^\mu \times 1, i\sqrt{2}\gamma_5 \alpha^m)$$

where  $(a^5, a^6) = (a, a^\dagger)$  and

$$[a, a^\dagger] = 1 \quad (F.1)$$

Then the action of  $\Gamma^\mu$  is given by  $\langle i | \Gamma^\mu | j \rangle = \gamma^\mu \eta_{ij}$  where  $\eta_{ij} = \langle i | j \rangle$  and we choose

$$|\eta_{ij}| = \delta_{ij} \quad (F.2)$$

To find  $\langle i | \Gamma^M | j \rangle$  we need to calculate  $\langle i | a^M | j \rangle$ .

We take the action of  $a^m$  on  $|i\rangle$  to be

$$a|i\rangle = q|i-1\rangle, a^\dagger|i\rangle = q'|i+1\rangle. \quad (F.3)$$

where  $q$  and  $q'$  are appropriate real normalisation factors. Normally one chooses a state  $|0\rangle$  to be annihilated by  $a$  i.e.  $a|0\rangle = 0$ . In our formalism we have chosen a state  $|c\rangle$  satisfying  $\Sigma_{mn}|c\rangle = 0$ . Since  $\Sigma_{mn} = -\{\Gamma_m, \Gamma_n\}/4$  we have  $(a^\dagger a + \frac{1}{2})|c\rangle = 0$ . Taking  $a^\dagger a|c\rangle = c|c\rangle$  this implies  $c = -\frac{1}{2}$  so the set of states is infinite dimensional. Then we have

$$a^\dagger a |-\frac{1}{2}\rangle = -\frac{1}{2} \quad (F.4)$$

and we let

$$\langle -\frac{1}{2} | -\frac{1}{2} \rangle = 1. \quad (F.5)$$

From these relations we can now construct  $\langle i | a^M | j \rangle$ .

First consider

$$a |-\frac{1}{2}\rangle = q |-\frac{3}{2}\rangle$$

This gives us

$$\langle -\frac{1}{2} | a^\dagger a |-\frac{1}{2}\rangle = q^2 \langle -\frac{3}{2} | -\frac{3}{2} \rangle$$

By (F.4) and (F.5) this becomes

$$-\frac{1}{2} = q^2 \langle -3/2 | -3/2 \rangle$$

Since  $q$  is real we have  $\langle -3/2 | -3/2 \rangle = -1$  and we take  $q = +1/\sqrt{2}$ .

In general let  $a^n | -\frac{1}{2} \rangle = q_n | -n-\frac{1}{2} \rangle$  then

$$\langle -\frac{1}{2} | (a^\dagger)^n a^n | -\frac{1}{2} \rangle = q_n^2 \langle -n-\frac{1}{2} | -n-\frac{1}{2} \rangle \quad (\text{F.6})$$

and we obtain

$$\langle -\frac{1}{2} | (a^\dagger)^{n+1} a^{n+1} | -\frac{1}{2} \rangle = (-1)^{n+1} (n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2}) \dots (\frac{1}{2}); n \geq 0 \quad (\text{F.7})$$

In order to prove the above result we need

$$a^\dagger a^n = a^n a^\dagger - n a^{n-1} \quad (\text{F.8})$$

[This can easily be proved by induction. It is obviously true for  $n=0$ . If we assume it is true for some  $N$  then

$$a^\dagger a^{N+1} = (a^\dagger a^N) a = a^N a^\dagger a - N a^N = -(N+1) a^N + a^{N+1} a^\dagger$$

by (F.1). Hence the statement is true  $\forall n \geq 0$ .]

The proof of (F.7) is by induction. It is true for  $n=0$  by (F.4) and (F.5). If we assume it is true for some  $N$  then

$$\begin{aligned} \langle -\frac{1}{2} | (a^\dagger)^{N+1} a^{N+1} | -\frac{1}{2} \rangle &= \langle -\frac{1}{2} | (a^\dagger)^N (a^N a^\dagger - N a^{N-1}) a | -\frac{1}{2} \rangle \\ &= -(N+\frac{1}{2}) \langle -\frac{1}{2} | (a^\dagger)^N a^N | -\frac{1}{2} \rangle \text{ by (C.4).} \end{aligned}$$

Hence it is true for all  $n \geq 0$ .

Now (F.6) and (F.7) imply that

$$a^n | -\frac{1}{2} \rangle = [(n-\frac{1}{2})(\frac{1}{2})]^{1/2} | -n-\frac{1}{2} \rangle; n \geq 0 \quad (\text{F.9})$$

$$\langle -n-\frac{1}{2} | -n-\frac{1}{2} \rangle = (-1)^n; n > 0 \quad (\text{F.10})$$

and (F.9) implies

$$a | -n-\frac{1}{2} \rangle = (n+\frac{1}{2})^{1/2} | -n-1-\frac{1}{2} \rangle; n \geq 0 \quad (\text{F.11})$$

In exactly the same way we have for  $a^\dagger$ ,  $(a^\dagger)^n | -\frac{1}{2} \rangle = q'_n | n-\frac{1}{2} \rangle$

$$\langle -\frac{1}{2} | a^n (a^\dagger)^n | -\frac{1}{2} \rangle = q'^n_n \langle n-\frac{1}{2} | n-\frac{1}{2} \rangle \quad (\text{F.12})$$

$$\langle -\frac{1}{2} | a^{n+1} (a^\dagger)^{n+1} | -\frac{1}{2} \rangle = (n+\frac{1}{2})(n-\frac{1}{2})(n-3/2) \dots (\frac{1}{2}); n \geq 0 \quad (\text{F.13})$$

where we use the result

$$a(a^\dagger)^n = n(a^\dagger)^{n-1} + (a^\dagger)^n a \quad (\text{F.14})$$

and (F.13) and (F.14) are proved by induction as before.

Then (F.12) and (F.13) imply that

$$(a^\dagger)^n |-\frac{1}{2}\rangle = [(n-\frac{1}{2}) \cdot (\frac{1}{2})]^{\frac{1}{2}} |n-\frac{1}{2}\rangle; n \geq 0 \quad (\text{F.15})$$

$$\langle n-\frac{1}{2} | n-\frac{1}{2} \rangle = 1; n \geq 0 \quad (\text{F.16})$$

and (F.15) implies

$$a^\dagger |n-\frac{1}{2}\rangle = (n+\frac{1}{2})^{\frac{1}{2}} |n+1-\frac{1}{2}\rangle; n \geq 0. \quad (\text{F.17})$$

We can now calculate  $a |n-\frac{1}{2}\rangle; n > 0$  by

$$\begin{aligned} a |n-\frac{1}{2}\rangle &= a(a^\dagger)^n |-\frac{1}{2}\rangle [(\frac{1}{2}) \cdot (n-\frac{1}{2})]^{-\frac{1}{2}}; n \geq 0 \\ &= (n(a^\dagger)^{n-1} + (a^\dagger)^n a) |-\frac{1}{2}\rangle [(\frac{1}{2}) \cdot (n-\frac{1}{2})]^{-\frac{1}{2}}; n \geq 0 \text{ by (F.14)} \\ &= (n-\frac{1}{2})(a^\dagger)^{n-1} |-\frac{1}{2}\rangle [(\frac{1}{2}) \cdot (n-\frac{1}{2})]^{-\frac{1}{2}}; n \geq 0 \text{ by (F.4)} \\ &= (n-\frac{1}{2}) [(\frac{1}{2}) \cdot (n-1-\frac{1}{2})]^{-\frac{1}{2}} [(\frac{1}{2}) \cdot (n-\frac{1}{2})]^{-\frac{1}{2}} |n-1-\frac{1}{2}\rangle; n > 0 \text{ by (F.15)} \\ &= (n-\frac{1}{2})^{\frac{1}{2}} |n-1-\frac{1}{2}\rangle; n > 0 \end{aligned} \quad (\text{F.18})$$

Similarly, using (F.8), (F.4) and (F.9) we find that

$$a^\dagger | -n-\frac{1}{2} \rangle = -(n-\frac{1}{2})^{\frac{1}{2}} | -n+1-\frac{1}{2} \rangle \quad n \geq 1 \quad (\text{F.19})$$

Finally we have

$$\langle -m-\frac{1}{2} | a | -m+\frac{1}{2} \rangle = (-1)^m (m-\frac{1}{2})^{\frac{1}{2}} \text{ by (F.10) and (F.11)}$$

$$\langle n+\frac{1}{2} | a^\dagger | n-\frac{1}{2} \rangle = (n+\frac{1}{2})^{\frac{1}{2}} \text{ by (F.16) and (F.17)}$$

$$\langle n-\frac{1}{2} | a | n+\frac{1}{2} \rangle = (n+\frac{1}{2})^{\frac{1}{2}} \text{ by (F.16) and (F.18)}$$

$$\langle -m+\frac{1}{2} | a^\dagger | -m-\frac{1}{2} \rangle = (-1)^m (m-\frac{1}{2})^{\frac{1}{2}} \text{ by (F.10) and (F.19)}$$

for  $n \geq 0$  and  $m > 0$  and

$$\eta_{|-\frac{1}{2}\rangle, |-\frac{1}{2}\rangle} = (-1)^{(|\uparrow| - |\downarrow|)/2}$$

by (F.10) and (F.16).



## APPENDIX G : GRASSMANN STATES AND THE GRASSMANN OSCILLATOR

In this appendix we investigate Grassmann states and the Grassmann oscillator. It will be helpful, before we begin the fermionic case to revise the bosonic case and then proceed by analogy.

We begin by looking at a one dimensional bosonic oscillator.

In the energy basis we represent the states of the system by  $|n\rangle$  where the energy of the  $n$ th level is given by  $E_n = (n + \frac{1}{2})$  (in units of  $\hbar$ ). These states are orthogonal and normalised i.e.  $\langle n|n'\rangle = \delta_{nn'}$  and we have a completeness relation  $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$ . The algebra of the system is given by  $[a, a^\dagger] = 1$  where  $a^\dagger$  creates a quantum of unit energy and  $a$  annihilates such a quantum. It is sometimes useful to consider a representation in which the states are labelled instead by an eigenvalue of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

These states are simply a linear superposition of the  $|n\rangle$  states

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n |n\rangle / (n!)^{\frac{1}{2}} = e^{a^\dagger \alpha} |0\rangle$$

and the adjoint state is given by

$$\langle\alpha| = \sum_{n=0}^{\infty} \langle n| (\alpha^*)^n / (n!)^{\frac{1}{2}} = \langle 0| e^{\alpha^* a}$$

The  $|\alpha\rangle$  states are neither orthogonal nor normalised since

$$\begin{aligned} \langle\alpha|\alpha'\rangle &= \sum_{n,n'} (\alpha^*)^n (\alpha')^{n'} \langle n|n'\rangle / (n!n')^{\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} (\alpha^*)^n (\alpha')^n / n! \\ &= e^{\alpha^* \alpha'} \end{aligned}$$

To find the appropriate completeness relation we consider the scalar product of two arbitrary states

$$\begin{aligned}
\langle \psi | \psi' \rangle &= \sum_{n=0}^{\infty} \langle \psi | n \rangle \langle n | \psi' \rangle \\
&= \int_0^{\infty} d|\alpha|^2 \sum_{n=0}^{\infty} \psi_n^* \psi'_n |\alpha|^{2n} e^{-|\alpha|^2} / n! \\
&= \int d^2\alpha \delta_{nn'} \left( \sum_{n'=0}^{\infty} \alpha^{n'} \psi_n^* / (n'!)^{\frac{1}{2}} \right) \left( \sum_{n=0}^{\infty} \alpha^{*n} \psi'_n / (n!)^{\frac{1}{2}} \right) e^{-|\alpha|^2} / \pi \\
&= \int d^2\alpha \delta_{nn'} \langle \psi | \alpha \rangle \langle \alpha | \psi' \rangle e^{-|\alpha|^2} / \pi
\end{aligned}$$

where we have used  $n! = \int dz z^n e^{-z}$  for  $z$  a commuting number.

So the completeness relation for the  $|\alpha\rangle$  states is

$$1 = \int d^2\alpha (|\alpha\rangle e^{-|\alpha|^2} \langle \alpha|) / \pi$$

Now let us find the analogous results for a one dimensional fermionic oscillator. The algebra is given by  $\{Q, Q^\dagger\} = 1$  where  $Q^\dagger$  creates one unit of fermion number and  $Q$  destroys it. Now  $Q^{\dagger 2} = 0$  so we only have two states,  $|0\rangle$  which commutes and  $|1\rangle = Q^\dagger |0\rangle$  which must anti-commute. Changing representation we have

$$Q|q\rangle = q|q\rangle$$

where the state  $|q\rangle$  commutes and is given by a linear superposition over the  $|n\rangle$  states

$$|q\rangle = \sum_{n=0}^1 q^n |n\rangle = e^{Q^\dagger q} |0\rangle = |0\rangle - q|1\rangle$$

The adjoint state is

$$\langle q| = \langle 0| e^{q^* Q} = \langle 0| + q^* \langle 1|$$

since  $q^*$  anticommutes.

The states  $|q\rangle$  are not orthogonal or normalised since

$$\langle q|q'\rangle = (\langle 0| + q^* \langle 1|)(|0\rangle - q'|1\rangle) = 1 + q^* q' = e^{q^* q'}$$

Their completeness relation is found by taking the scalar product of two arbitrary states

$$\begin{aligned}
\langle \psi | \psi' \rangle &= \sum_{n=0}^1 \langle \psi | n \rangle \langle n | \psi' \rangle \\
&= \sum_{n=0}^1 \int dq^* (q^*)^n e^{-q^*} \psi_n^* \int dq(q)^n e^{-q} \psi'_n
\end{aligned}$$

since  $\int dz e^{-z} = -1$  and  $\int dz z e^{-z} = 1$  for  $z = (q, q^*)$ . Hence

$$\begin{aligned}
\langle \psi | \psi' \rangle &= \int dq^* dq e^{-q^* q} (\psi_0^* \psi'_0 + q \psi_1^* q^* \psi'_1) \\
&= \int dq^* dq e^{-q^* q} (\langle \psi | q \rangle \langle q | \psi' \rangle)
\end{aligned}$$

since  $\langle q | \psi' \rangle = \sum_{n=0}^1 \langle q | n \rangle \langle n | \psi' \rangle = \langle q | 0 \rangle \psi'_0 + \langle q | 1 \rangle \psi'_1 = \psi'_0 + q^* \psi'_1$

and similarly for  $\langle \psi | q \rangle$ . Hence the completeness relation for the  $|q\rangle$  states is

$$1 = \int dq^* dq e^{-q^* q} |q\rangle \langle q|$$

Now in the bosonic case it is a simple matter to verify that the value of the observable  $H = a^\dagger a + \frac{1}{2}$ , corresponding to the energy of the system, cannot be negative. For our fermionic case we similarly define an observable  $H = Q^\dagger Q$ . Now let us look at the expectation value of this quantity

$$\langle q | H | q \rangle = \langle q | Q^\dagger Q | q \rangle = q^* q$$

The result is a nilpotent number which is unacceptable for a physical observable. At this stage we recall that in the bosonic case we are led to unnormalisable states by the consideration of pure eigenstates of momentum. To obtain sensible results in that situation it is necessary to work with wave packets centred on some average value of momentum. Let us look at a super-position of Grassmann states and see if this too can lead to sensible results.

In the one dimensional case let us consider a state

$$|c\rangle = (|0\rangle + c|1\rangle)/\sqrt{(1+|c|^2)}$$

The state is normalised and  $c$  is a commuting number. Then we can write  $|c\rangle$  as a super-position of the  $|q\rangle$  states

$$|c\rangle = \int dq^* dq e^{-q^* q} |q\rangle \langle q|c\rangle$$

where

$$\langle q|c\rangle = e^{q^* c}/\sqrt{(1+|c|^2)}$$

Taking the expectation of  $H$  between two  $|c\rangle$  states we find

$$\langle c|H|c\rangle = |c|^2/\sqrt{(1+|c|^2)}$$

which is an acceptable result.

Provided we are willing to accept states which are polynomials in Grassmann variables as our physical states and we follow the conventions of Grassmann integration we are led to no inconsistencies in this extension of Dirac's formalism.

So far we have only discussed the one dimensional case but it is quite simple to extend this to  $N$  dimensions.

For an  $N$ -dimensional Grassmann oscillator we have a set of  $N$  creation and annihilation operators obeying the algebra

$$\{Q_i, Q_j^\dagger\} = \delta_{ij}$$

We again construct states labelled by the eigenvalues of the  $Q_i$

$$Q_i |q\rangle = q_i |q\rangle \tag{G.1}$$

where  $|q\rangle$  is given by

$$|q\rangle = e^{Q_i^\dagger q_i} |0\rangle = |0\rangle - q_i |i\rangle + \frac{1}{2} q_i q_j |ij\rangle + \frac{1}{6} q_i q_j q_k |ijk\rangle + \dots$$

We have  $|ij\rangle = Q_i^\dagger Q_j^\dagger |0\rangle = -|ji\rangle$  and the sign factors in the above

expansion are due to the anti-commutation of  $Q_i^\dagger$  and  $q_j$ .

As a check

$$\begin{aligned} Q_k |q\rangle &= 0 + q_i \delta_k^i |0\rangle - \frac{1}{2} q_i q_j \delta_k^i |j\rangle + \frac{1}{2} q_i q_j \delta_k^j |i\rangle + \dots \\ &= q_k (|0\rangle - q_i |i\rangle + \dots) \end{aligned}$$

So  $|q\rangle$  does satisfy (6.1)

The completeness relation can be obtained as before by taking the scalar product of two arbitrary states. It is found to be

$$1 = \int d^N q^* d^N q e^{-q^* i q_i} |q\rangle \langle q|$$

Physical states should then be written as a super-position over the  $|q\rangle$ .

$$|c\rangle = \int d^N q^* d^N q e^{-q^* i q_i} |q\rangle \langle q| c\rangle = |0\rangle - c_i |i\rangle - c_i c_j |ij\rangle + \dots$$

with the  $c_i$  commuting numbers and

$$\langle q|c\rangle = e^{q^* i c_i}$$

In conclusion, we have found that it is possible to consistently generalise Dirac's notation to incorporate Grassmann operators and states.